QUIZ 9

PRINT YOUR **FULL** NAME: ____________________

1. (4 points) Decide whether these are true or false.(circle **T/F**)

   - **T**/**F** The number of (nonzero) singular values of a m × n matrix is smaller than m and n.

     no more than

     # singular values = min(m,n)
   - **T**/**F** If A is n×m and P is a n×n orthogonal matrix then A and PA have the same singular values.

     Homework problem
   - **T**/**F** The singular values of a square matrix are equal to its eigenvalues. ← no direct relation
   - **T**/**F** The singular values of a square symmetric matrix A are the square roots of its eigenvalues.

     $|\lambda_i| = \sigma_i$. λ_i: eigenvalue, σ_i: singular values.

2. (2 points) Orthogonally diagonalize the matrix:

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

sol: Step 1: find eigenvalues.

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = 0$$

$\lambda^2 - 6\lambda + 8 = 0$

$(\lambda-2)(\lambda-4) = 0$

$\Rightarrow \lambda_1 = 4,\ \lambda_2 = 2$

Step 2: find eigenvectors

$(A - 4I)x = 0 \quad \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} x = 0$

sol. set $\left\{ x = \begin{pmatrix} 1 \\ -1 \end{pmatrix} s : s \in \mathbb{R} \right\}$

$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(A - 2I)x = 0 \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} x = 0$

sol. set $\left\{ x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} s : s \in \mathbb{R} \right\}$

$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Step 3: normalize it

$u_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$u_2 = \dfrac{v_2}{\|v_2\|} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$P = (u_1\ u_2) = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$D = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$

3. (3 points) For the following matrix,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

   (1) find all singular values,
   (2) find the SVD.

Sol: (1) $A^TA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$\det(A^TA - \lambda I) = (1-\lambda)\left[(1-\lambda)^2 - 1\right] = (1-\lambda)\lambda(\lambda - 2) = 0$

$\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = 0$

singular value of A = min(2,3) = 2

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{2}, \quad \sigma_2 = \sqrt{\lambda_2} = 1$

(2) $(A^TA - 2I)x = 0$

$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} x = 0 \quad$ sol set: $\left\{ x = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} s : s \in \mathbb{R} \right\}$

$$v_1 = \frac{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$(A^TA - I)\, x = 0$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} x = 0 \qquad \text{sol set: } \left\{ x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} s \;:\; s \in \mathbb{R} \right\}$$

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \|v_2\| = 1 \text{ (no need of normalizing)}$$

$$(A^TA - 0I)\, x = 0$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} x = 0 \qquad \text{sol. set: } \left\{ x = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} s \;:\; s \in \mathbb{R} \right\}$$

$$v_3 = \frac{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}{\left\| \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad V = (v_1\ v_2\ v_3) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$U = (u_1\ u_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A = U \Sigma V^T$$

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Alternative solution:

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \qquad V = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$