HANDOUT 9 (REVIEW FOR MIDTERM 1)

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- How to solve systems of linear equations?
  Write down the augmented matrix and do row reduction to obtain RREF.
  The solutions: (1) unique solution, (2) infinite solution, (3) no solution.
  consistent: either (1) or (2)
- How to solve vector equation? For example, $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$.
  Same as before.
- How to solve matrix equation? For example, $A\mathbf{x} = \mathbf{b}$.
  Same as before.

- How to determine whether $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$?
  Solve vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$.
  If consistent, $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Otherwise, No.
  - How to determine the system is consistent given the echelon form of the augmented matrix?
  - How to determine the system has unique solution given the echelon form of the augmented matrix?
- How to determine whether $\mathbf{b}$ is in $span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?
  Same as before
- How to determine $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent?
  Check whether vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has only trivial solution.
  If unique solution, linearly independent, otherwise (infinite solution), linearly dependent.
- How to determine a transformation to be a linear transformation?
  Verify that
  (1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$.
  (2) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ and all $c \in \mathbb{R}$.
- How to determine a linear transformation $T(\mathbf{x}) = A\mathbf{x}$ to be a one-to-one mapping?
  Solve matrix equation $A\mathbf{x} = 0$
  If unique solution, T is one-to-one mapping, otherwise (infinite solution), No.
- How to determine a linear transformation $T(\mathbf{x}) = A\mathbf{x}$ to be an onto mapping?
  Check whether matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent for any $\mathbf{b}$?
  If always consistent, T is an onto. Otherwise, No.
- How to calculate the $\mathcal{B}$-coordinate vector of $\mathbf{b}$ given that the vector $\mathbf{b}$ is in a subspace H with a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
  Solve vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$

Some variants of problems:

1. Determine the value(s) of h such that the following is the augmented matrix of a consistent linear system:

$$\left(\begin{array}{cc|c} h & 1 & -2 \\ 4 & h & 4 \end{array}\right)$$

2. Find an equation involving g, h that makes this augmented matrix corresponding to a consistent system:

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & 0 \\ -2 & 5 & -9 & h \end{bmatrix}$$

3. Find all $\mathbf{x} \in \mathbb{R}^3$ that are mapped into the zero vector by the transformation $\mathbf{x} \mapsto A\mathbf{x}$ for the given matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 4 & 6 \end{bmatrix}$.

5. Let T be a linear transform with $T\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}, T\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}$. Determine the matrix A_T of T.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ to $\begin{bmatrix}4\\5\end{bmatrix}$ and maps $\mathbf{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ to $\begin{bmatrix}2\\3\end{bmatrix}$. use the fact that T is linear to find the images under T of $\begin{bmatrix}5\\0\end{bmatrix}, \begin{bmatrix}0\\3\end{bmatrix}, \begin{bmatrix}5\\3\end{bmatrix}$.

6. Find the point (x_1, x_2) that lies on the line $x_1 + 5x_2 = 7$ and the line $x_1 - 2x_2 = -2$.

7. Let $\mathbf{v}_1 = \begin{bmatrix}1\\-2\\-3\end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix}1\\6\\5\end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix}2\\4\\2\end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix}5\\6\\h\end{bmatrix}$. For what value of h is $\mathbf{u}$ in the plane spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$?

8. Let $\mathbf{v}_1 = \begin{bmatrix}1\\-2\\-3\end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix}1\\6\\5\end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix}2\\4\\2\end{bmatrix}$. Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$?

9. Given $A = \begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix}$, observe that the first column plus the twice the second column equals the third column. Find a nontrivial solution $A\mathbf{x} = \mathbf{0}$.

Some Takeaway

- If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and $\mathbf{v}_3$ is not in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
- $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is different from Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
- The parametric equation of the line through $\mathbf{a}$ parallel to $\mathbf{b}$: $\mathbf{x} = \mathbf{a} + t\mathbf{b}$.
- The parametric equation of the line through $\mathbf{a}$ and $\mathbf{b}$: $\mathbf{x} = t\mathbf{a} + (1-t)\mathbf{b}$.