HANDOUT 5

JIASU WANG

1. Please explain the following terms: domain, codomain, image, range, onto (surjective), one-to-one(injective), bijective, linear transformation.

2. Write down the standard matrix for the following transformation

(1) reflection through x axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(2) reflection through y axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(3) reflection through $y = x$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(4) reflection through $y = -x$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

(5) reflection through the origin

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

(6) horizontal contraction and expansion

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

(7) vertical contraction and expansion

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

(8) horizontal shear (shear factor is 2)

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

(9) vertical shear (shear factor is 3)

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

(10) projection onto x axis

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(11) projection onto y axis

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(12) counterclockwise rotation about the origin through 90°

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and maps $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 6 \end{bmatrix}$. use the fact that T is linear to find the images under T of $3\mathbf{u}$, $2\mathbf{v}$ and $3\mathbf{u}+2\mathbf{v}$.

Sol: $T(3\mathbf{u}) = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$, $T(2\mathbf{v}) = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$, $T(3\mathbf{u}+2\mathbf{v}) = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$,

4. Find all $\mathbf{x} \in \mathbb{R}^3$ that are mapped into the zero vector by the transformation $\mathbf{x} \mapsto A\mathbf{x}$ for the given matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 4 & 6 \end{bmatrix}$. (The set of all vectors whose image is zero vector is called **kernel**)

Sol: $\{(x_1, x_2, x_3) = (5s, -4s, s) : s \in \mathbb{R}\}$

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ with $T(\mathbf{x}) = A\mathbf{x}$ for the given matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ -2 & 5 & -4 \\ 4 & 5 & -3 \end{bmatrix}$.

(1) Determine if T is a one-to-one mapping.
(2) Determine if T is an onto mapping.

Sol: (1) Yes, (2) Yes

(**Takeaway**) Given linear transformation $T : x \mapsto Ax$,

(1) T is a one-to-one mapping if and only if $Ax = 0$ has only trivial solution.
(2) T is an onto mapping if and only if for any b, $Ax = b$ always has solution.

(**Takeaway**) Given linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$,

(1) If $n = m$, then T is a one-to-one mapping if and only if T is an onto mapping.
(2) If T is a one-to-one mapping, then $n \leq m$.
(3) If T is an onto mapping, then $n \geq m$.