HANDOUT 4

JIASU WANG

1. Please explain the following term: Linearly independent.
2. Determine if the vectors are linearly independent.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix},$$

Sol: $\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}_3$. These vectors are linearly dependent.

3.(Extra exercise) Find linearly independent row vectors and column vectors of

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Sol:

For matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, the linearly independent row vectors are $[1, 2]$ and $[3, 4]$. The linearly independent column vectors are $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

For matrix, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, the linearly independent row vectors are $[1, 2, 3]$ and $[4, 5, 6]$. The linearly independent column vectors are $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$.

4. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is called a **shear transformation**. It can be shown that if T acts on each point in the 2×2 square, then the set of images forms the sheared parallelogram. The key idea is to show that T maps line segments onto line segments and then to check that the corners of the square map onto the vertices of the parallelogram.

(**Takeaway**) For horizontal shear transformation with shear factor k, the matrix is $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. For vertical shear transformation with shear factor k, the matrix is $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

5. Define a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

(1) Show that $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$, where $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(2) T rotates $\mathbf{u}, \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ counterclockwise about the origin through $90°$, This is one of the **rotation transformation**.

(3) (**Takeaway**) The matrix A for rotation transformation must take the following form:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Here $\theta \in \mathbb{R}$. It represents the counterclockwise rotation about the origin through θ.