Math 54 Worksheet

Constant Coefficient Linear Systems

Part A

These questions test your knowledge of the core concepts and computations.

1. Consider the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with

$$\mathbf{A} = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$

   (a) Show that the matrix $\mathbf{A}$ has eigenvalues $r_1 = 2$ and $r_2 = -2$ with corresponding eigenvectors $\mathbf{u}_1 = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$.

   (b) Give a general solution to this linear system

Solution:

(a)

$$\det(A - rI_2) = \det \begin{bmatrix} 1-r & \sqrt{3} \\ \sqrt{3} & -1-r \end{bmatrix}$$

$$(1-r)(-1-r) - 3 = 0$$

$$\implies r^2 = 4 \implies r = \pm 2$$

For $r = 2$,

$$\text{Nul} \begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{bmatrix} = \text{Span}\left(\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}\right)$$

For $r = -2$,

$$\text{Nul} \begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \text{Span}\left(\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}\right)$$

(b) A general solution to the system is $\mathbf{x}(t) = c_1 \begin{bmatrix} \sqrt{3}e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-2t} \\ -\sqrt{3}e^{-2t} \end{bmatrix}$

2. Use the substitution $x_1 = y, x_2 = y'$ to convert the linear equation $ay'' + by' + cy = 0$, where a, b, and c are constants, into a normal system. Show that the characteristic equation for this system is the same as the auxiliary equation for the original equation.
Solution:

$$ay'' + by' + cy = 0$$

$$\implies y'' = \frac{-by' - cy}{a} = \frac{-bx_2 - cx_1}{a}$$

$$\implies \mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-bx_2 - cx_1}{a} \end{bmatrix}$$

$$\implies x_1 \begin{bmatrix} 0 \\ -c/a \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -b/a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix} \mathbf{x} = A\mathbf{x}$$

$$\det(A - \lambda I_2) = (-\lambda)(-b/a - \lambda) + c/a = 0$$

$$\implies \lambda^2 + b/a\lambda + c/a = 0$$

$$\implies a\lambda^2 + b\lambda + c = 0$$

Thus, the characteristic equation for this system is the same as the equation $ay'' + by' + cy = 0$.

Part B

These questions are generally more challenging, often highlighting important subtleties. They require a deeper understanding of each concept and the interrelations between them.

3. Find a general solution of the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ for the given matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix}$$

Solution:
We can find the eigenvalues of A to be $\lambda_1 = 2i$ and $\lambda_2 = -2i$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$. Use the following equations:

$$\mathbf{x}_1(t) := e^{\alpha t}\cos\beta t\mathbf{a} - e^{\alpha t}\sin\beta t\mathbf{b}$$
$$\mathbf{x}_2(t) := e^{\alpha t}\sin\beta t\mathbf{a} + e^{\alpha t}\cos\beta t\mathbf{b}$$

Here, $\alpha = 1, \beta = 1, \mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, we have the following:

$$\mathbf{x}_1(t) := e^t\cos t\begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^t\sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^t\begin{bmatrix} \cos t - \sin t \\ \cos t \end{bmatrix}$$
$$\mathbf{x}_2(t) := e^t\sin t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^t\cos t\begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^t\begin{bmatrix} \sin t + \cos t \\ \sin t \end{bmatrix}$$

Thus, we have:

$$\mathbf{x}(t) = c_1e^t\begin{bmatrix} \cos t - \sin t \\ \cos t \end{bmatrix} + c_2e^t\begin{bmatrix} \sin t + \cos t \\ \sin t \end{bmatrix}$$

4. Show that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ given by equations

$$\mathbf{x}_1(t) := e^{\alpha t}\cos\beta t\mathbf{a} - e^{\alpha t}\sin\beta t\mathbf{b}$$
$$\mathbf{x}_2(t) := e^{\alpha t}\sin\beta t\mathbf{a} + e^{\alpha t}\cos\beta t\mathbf{b}$$

can be obtained as linear combinations of $\mathbf{w}_1(t)$ and $\mathbf{w}_2(t)$ given by equations

$$\mathbf{w}_1(t) = e^{r_1 t}\mathbf{z} = e^{(\alpha+i\beta)t}(\mathbf{a}+i\mathbf{b})$$
$$\mathbf{w}_2(t) = e^{r_2 t}\overline{\mathbf{z}} = e^{(\alpha-i\beta)t}(\mathbf{a}-i\mathbf{b})$$

Note that $\alpha + i\beta$ and $\mathbf{a} + i\mathbf{b}$ are eigenvalue and eigenvector for A.

Solution:

$$\mathbf{w}_1(t) = e^{(\alpha+i\beta)t}(\mathbf{a}+i\mathbf{b}) = e^{\alpha t}e^{i\beta t}(\mathbf{a}+i\mathbf{b})$$

Note that $e^{i\beta t} = \cos\beta t - i\sin\beta t$:

$$\mathbf{w}_1(t) = e^{\alpha t}(\cos\beta t - i\sin\beta t)(\mathbf{a}+i\mathbf{b})$$

$$\mathbf{w}_1(t) = e^{\alpha t}\cos\beta t\mathbf{a} + ie^{\alpha t}\sin\beta t\mathbf{a} + ie^{\alpha t}\cos\beta t\mathbf{b} - e^{\alpha t}\sin\beta t\mathbf{b}$$

Similarly, we have:

$$\mathbf{w}_2(t) = e^{\alpha t}\cos\beta t\mathbf{a} - ie^{\alpha t}\sin\beta t\mathbf{a} - ie^{\alpha t}\cos\beta t\mathbf{b} - e^{\alpha t}\sin\beta t\mathbf{b}$$

With these two equations, it can be seen that:

$$\mathbf{x}_1(t) = \frac{\mathbf{w}_1(t) + \mathbf{w}_2(t)}{2}$$

$$\mathbf{x}_2(t) = \frac{i(\mathbf{w}_2(t) - \mathbf{w}_1(t))}{2}$$

