Math 54 Worksheet

Constant Coefficient Linear Systems

Part A

These questions test your knowledge of the core concepts and computations.

1. Consider the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with

$$\mathbf{A} = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$

   (a) Show that the matrix $\mathbf{A}$ has eigenvalues $r_1 = 2$ and $r_2 = -2$ with corresponding eigenvectors $\mathbf{u}_1 = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$.

   (b) Give a general solution to this linear system

   Solution:

2. Use the substitution $x_1 = y, x_2 = y'$ to convert the linear equation $ay'' + by' + cy = 0$, where a, b, and c are constants, into a normal system. Show that the characteristic equation for this system is the same as the auxiliary equation for the original equation.
Solution:

Part B

These questions are generally more challenging, often highlighting important subtleties. They require a deeper understanding of each concept and the interrelations between them.

3. Find a general solution of the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ for the given matrix $\mathbf{A}$.

   1. $\mathbf{A} = \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix}$ 2. $\mathbf{A} = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$

   Solution:

4. Show that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ given by equations

$$\mathbf{x}_1(t) := e^{\alpha t}\cos\beta t\mathbf{a} - e^{\alpha t}\sin\beta t\mathbf{b}$$
$$\mathbf{x}_2(t) := e^{\alpha t}\sin\beta t\mathbf{a} + e^{\alpha t}\cos\beta t\mathbf{b}$$

can be obtained as linear combinations of $\mathbf{w}_1(t)$ and $\mathbf{w}_2(t)$ given by equations

$$\mathbf{w}_1(t) = e^{r_1 t}\mathbf{z} = e^{(\alpha+i\beta)t}(\mathbf{a}+i\mathbf{b})$$
$$\mathbf{w}_2(t) = e^{r_2 t}\overline{\mathbf{z}} = e^{(\alpha-i\beta)t}(\mathbf{a}-i\mathbf{b})$$

Note that $\alpha + i\beta$ and $\mathbf{a} + i\mathbf{b}$ are eigenvalue and eigenvector for A.

Solution: