Math 54 Worksheet

Linear Systems of Differential Equations

Part A

These questions test your knowledge of the core concepts and computations.

1. Transform the following system in to the matrix form.

$$\begin{aligned} dx/dt &= (t^3+2)x + 2z \\ dy/dt &= (e^t-1)y + (t-1)z \\ dz/dt &= (e^{2t}+e^{3t})x + (t+3)y \end{aligned}$$

Solution:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} t^3+2 & 0 & 2 \\ 0 & e^t-1 & t-1 \\ e^{2t}+e^{3t} & (t+3) & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

2. Write down the equivalent linear system of differential equations to the following differential equation.

$$y''''(t) + (3t-2)y'''(t) + (t+3)y'(t) - e^t y(t) = 0$$

Solution: let $y_0(t) = y(t), y_1(t) = y'(t), y_2(t) = y''(t), y_3(t) = y'''(t)$. Then we have

$$\frac{d}{dt}\begin{bmatrix} y_0(t) \\ y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ e^t & -t-3 & 0 & 2-3t \end{bmatrix}\begin{bmatrix} y_0(t) \\ y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

3. Consider the following system of second order differential equations with m and k being positive real numbers:

$$\begin{cases} mx_1'' &= -2kx_1 + kx_2 \\ mx_2'' &= kx_1 - 2kx_2 \end{cases}$$

   (a) Using the substitution $y_i = mx_i'$ for $i = 1, 2$ to tranform the system into a system of first-order differential equation

   (b) Write down the matrix form of the equation.

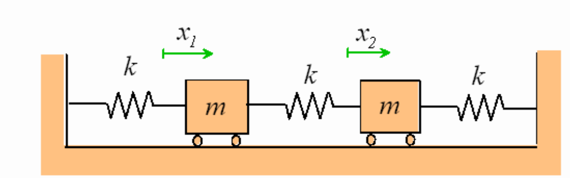
Solution: When we make the substitution, we get $y_i' = mx_i''$, hence we have the following matrix form equation

$$\begin{cases} x_1' &= \frac{1}{m}y_1 \\ x_2' &= \frac{1}{m}y_2 \\ y_1' &= -2kx_1 + kx_2 \\ y_2' &= kx_1 - 2kx_2 \end{cases}$$

In matrix form this equation is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{m} & 0 \\ 0 & 0 & 0 & \frac{1}{m} \\ -2k & k & 0 & 0 \\ k & -2k & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ y_1(t) \\ y_2(t) \end{bmatrix}$$

(Aside: this system of equation describes the motion of the spring-mass system shown below with $x_1 = 0$ and $x_2 = 0$ corresponding to the neutral position.



Part B

These questions are generally more challenging, often highlighting important subtleties. They require a deeper understanding of each concept and the interrelations between them.

4. Decide whether these are true or false

T F Let $A(t)$ be an $n \times n$ matrix valued function, $g(t)$ is an $n-$dimensional real valued function. Assume $y_p(t)$ is a particular solution to $x'(t) = A(t)x(t) + g(t)$. Then all solutions of $x'(t) = A(t)x(t) + g(t)$ has the form $x_h(t) + p(t)$, where $x_h(t)$ is the solution to the homogeneous problem $x'(t) = A(t)x(t)$.

T F let $x_1(t), x_2(t), \dots, x_n(t)$ be solutions to the linear system of differential equations $x'(t) = A(t)x(t)$, Let $W(t) = det(x_1(t), \dots, x_n(t))$ be the wronskian. If $x_1(t), x_2(t), \dots, x_n(t)$ are linear independent, then $W(t) \neq 0$ for any $t \in \mathbb{R}$

T F If an $n \times n$ matrix A is not diagonalizable, then the dimension of the solution space of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is strictly less than n.

Solution:

(a) True, see lecture notes.

(b) True, see lecture notes.

(c) False, see problem 5 as an counterexample.

5. Consider the following system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

(a) Verify that $\begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} te^t \\ e^t \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2e^t \end{bmatrix}$ are all solutions to the above system.

(b) Determine if $\left\{ \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} te^t \\ e^t \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ e^{2t} \end{bmatrix} \right\}$ is a fundamental solution set.

Solution: We compute

$$\frac{d}{dt}\begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} te^t \\ e^t \\ 0 \end{bmatrix} = \begin{bmatrix} te^t + e^t \\ e^t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} te^t \\ e^t \\ 0 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} 0 \\ 0 \\ e^{2t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ e^{2t} \end{bmatrix}$$

The Wronskian is

$$\det \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} = e^{4t} \neq 0$$

So the collection is a fundamental solution set. Note that the matrix is upper triangular so the determinant is easy to compute

6. Consider the following system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Solve the above system by diagonalization. Write down the solutions you obtained and verify that they form a fundamental solution set by means of the Wronskian.

Solution:

We diagonalized the matrix before, this matrix has eigenvalues 1 and 4, with corresponding eigenspaces

$$E_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}; E_4 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\};$$

So we have solutions to the system $\begin{bmatrix} e^t \\ -e^t \\ 0 \end{bmatrix}, \begin{bmatrix} e^t \\ 0 \\ -e^t \end{bmatrix}, \begin{bmatrix} e^{4t} \\ e^{4t} \\ e^{4t} \end{bmatrix}$

We can plug the functions back in and see that they are indeed solutions. Now, to show that they are linearly independent, we considet the wronskian, which is the determinant of $\begin{bmatrix} e^t & e^t & e^{4t} \\ -e^t & 0 & e^{4t} \\ 0 & -e^t & e^{4t} \end{bmatrix}$, we cofactor expand along the first colum to find that

$$W(t) = e^t \left(0 - \left(-e^{5t}\right)\right) - \left(-e^t\right)\left(2e^{5t}\right) = 3e^{6t} \neq 0$$

So the solutions are linearly independent.