Math 54 Worksheet
Linear Systems of Differential Equations

Part A

These questions test your knowledge of the core concepts and computations.

1. Transform the following system in to the matrix form.

$$\begin{aligned} dx/dt &= (t^3 + 2)x + 2z \\ dy/dt &= (e^t - 1)y + (t - 1)z \\ dz/dt &= (e^{2t} + e^{3t})x + (t + 3)y \end{aligned}$$

Solution:

2. Write down the equivalent linear system of differential equations to the following differential equation.

$$y''''(t) + (3t-2)y'''(t) + (t+3)y'(t) - e^t y(t) = 0$$

Solution:

3. Consider the following system of second order differential equations with m and k being positive real numbers:

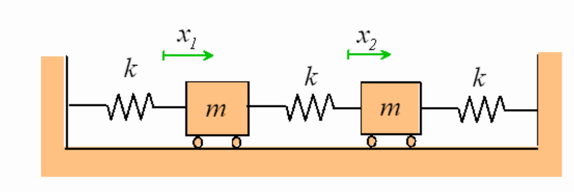
$$\begin{cases} mx_1''(t) &= -2kx_1(t) + kx_2(t) \\ mx_2''(t) &= kx_1(t) - 2kx_2(t) \end{cases}$$

   (a) Using the substitution $y_i = mx_i'$ for $i = 1, 2$ to transform the system into a system of first-order differential equation.

   (b) Write down the matrix form of the equation.

Solution:

(Aside: this system of equation describes the motion of the spring-mass system shown below with $x_1 = 0$ and $x_2 = 0$ corresponding to the neutral position.



Part B

These questions are generally more challenging, often highlighting important subtleties. They require a deeper understanding of each concept and the interrelations between them.

4. Decide whether these are true or false

T F Let $A(t)$ be an $n \times n$ matrix valued function, $g(t)$ is an $n-$dimensional real valued function. Assume $y_p(t)$ is a particular solution to $x'(t) = A(t)x(t) + g(t)$. Then all solutions of $x'(t) = A(t)x(t) + g(t)$ has the form $x_h(t) + p(t)$, where $x_h(t)$ is the solution to the homogeneous problem $x'(t) = A(t)x(t)$.

T F let $x_1(t), x_2(t), \dots, x_n(t)$ be solutions to the linear system of differential equations $x'(t) = A(t)x(t)$, Let $W(t) = det(x_1(t), \dots, x_n(t))$ be the wronskian. If $x_1(t), x_2(t), \dots, x_n(t)$ are linear independent, then $W(t) \neq 0$ for any $t \in \mathbb{R}$

T F If an $n \times n$ matrix A is not diagonalizable, then the dimension of the solution space of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is strictly less than n.

Solution:

5. Consider the following system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

(a) Verify that $\begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} te^t \\ e^t \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2e^t \end{bmatrix}$ are all solutions to the above system.

(b) Determine if $\left\{ \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} te^t \\ e^t \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ e^{2t} \end{bmatrix} \right\}$ is a fundamental solution set.

Solution:

6. Consider the following system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Solve the above system by diagonalization. Write down the solutions you obtained and verify that they form a fundamental solution set by means of the Wronskian.
Solution: