Math 54 Worksheet

Non-Homogeneous Second-Order Linear Differential Equations

Part A

These questions test your knowledge of the core concepts and computations.

1. Write a trial solution for the method of undetermined coefficients (Do not determine the coefficients)

1. $y'' - ey' + 2y = e^x + \sin x$

Homogeneous Part:

$r^2 - er + 2 = 0$

$r = \frac{e \pm \sqrt{e^2-8}}{2}$

$r_1 = \frac{e + \sqrt{e^2-8}}{2}$

$r_2 = \frac{e - \sqrt{e^2-8}}{2}$

$y_H = e^{\frac{e}{2}x}\left[C_1 \cos\left(\frac{\sqrt{e^2-8}}{2}\right)x + C_2 \sin\left(\frac{\sqrt{e^2-8}}{2}\right)x\right]$

Particular Solution:

$y_p = Ae^x + [B\cos x + C\sin x]$

General Solution:

$y = y_H + y_p$

$= e^{\frac{e}{2}x}\left[C_1 \cos\left(\frac{\sqrt{e^2-8}}{2}\right)x + C_2 \sin\left(\frac{\sqrt{e^2-8}}{2}\right)x + Ae^x + [B\cos x + C\sin x]\right]$

2. $y'' + 2y' + 10y = x^2 e^{-x}\cos 3x$

Homogeneous Part:

$r^2 + 2r + 10 = 0$

$r = -1 \pm 3i$

$\alpha = -1, \ \beta = 3$

Thus

$y_H = e^{-x}\left[C_1 \cos(3x) + C_2 \sin(3x)\right]$

Particular Solution:

$y_p = x(Ax^2+Bx+C)e^{-x}\cos 3x + x(Dx^2+Ex+F)e^{-x}\sin 3x$

Multiplied with x because of Repeated Root, y_H contained in y_p now

[y_p solution to y_H (Auxiliary Eq^n)]

General Solution:

$y = y_H + y_p$

$= x(Ax^2+Bx+C)e^{-x}\cos 3x + x(Dx^2+Ex+F)e^{-x}\sin 3x$

$=$

★ 3. $y'' - 6y' + 9y = e^{3x} + x\sin 2x$

Homogeneous Part:

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r = 3$

$r_1 = r_2 = 3$

Repeated Roots

$y_H = C_1 e^{3x} + C_2 x e^{3x}$

Particular Solution:

$y_p = x^2 A e^{3x} + (Bx + C)\sin 2x + (Dx + E)\cos 2x$

General Solution:

$y = y_H + y_p$

$= C_1 e^{3x} + C_2 x e^{3x} + x^2 A e^{3x} + (Bx+C)\sin 2x + (Dx+E)\cos 2x$

2. Solve the differential equations

   (a) $9y'' + y = e^{2x}$

   (b) $y'' - 4y' + 4y = x - \sin x$

   Solution:

Homogeneous Solution

$$9y'' + y = 0 \rightarrow 9r^2 + 1 = 0 \rightarrow r = \pm\sqrt{-\frac{1}{9}} = \pm\frac{1}{3}i$$

$$y_H = e^{\alpha x}\left[C_1 \cos(\beta x) + C_2 \sin(\beta x)\right]$$

$$y_H = \left[C_1 \cos\left(\tfrac{1}{3}\right)x + C_2 \sin\left(\tfrac{1}{3}\right)x\right]$$

Particular Solution:-

$$RHS = e^{2x} \leftarrow G(x)$$

$$y_p \rightarrow \text{Assume} \rightarrow \underline{\underline{Ae^{2x}}}$$

$$y_p' = 2Ae^{2x}$$

$$y_p'' = 4Ae^{2x}$$

$$\begin{aligned} LHS &= 9y'' + y \\ &= 9(4Ae^{2x}) + (Ae^{2x}) \\ &= 37Ae^{2x} \end{aligned} \quad \Bigg| \quad RHS = e^{2x}$$

Comparing LHS & RHS:

$$37Ae^{2x} = e^{2x}$$

$$37A = 1$$

$$\boxed{A = \frac{1}{37}}$$

Thus,

$$y_p = Ae^{2x} = \frac{1}{37}e^{2x}$$

General Solution:-

$$y = \underbrace{y_c} + \underbrace{y_p}$$

$$= \underbrace{\left[C_1 \cos\left(\tfrac{1}{3}\right)x + C_2 \sin\left(\tfrac{1}{3}\right)x\right]} + \underbrace{\left[\frac{1}{37}e^{2x}\right]}$$

2. Solve the differential equations

   (a) $9y'' + y = e^{2x}$

   (b) $y'' - 4y' + 4y = x - \sin x$

Solution:

Homogeneous Solution:

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2 \rightarrow$ Repeated form. (Homogeneous Solution)

$r_1 = 2$

$r_2 = 2$

$y_c = C_1 e^{r_1 x} + x C_2 e^{r_1 x}$ [Repeated Roots]

$y_c = C_1 e^{2x} + x C_2 e^{2x}$

Particular Solution

$y_p = (Ax+B) + (C\sin x + D\cos x)$

$y'_p = A + C\cos x - D\sin x$

$y''_p = -C\sin x - D\cos x$

Substituting back in main DE:

$y'' - 4y' + 4y = x - \sin x$

$[C\sin x - D\cos x] - 4[A + C\cos x - D\sin x] + 4[(Ax+B) + (C\sin x + D\cos x)] = x - \sin x$

$(4D + 3C)\sin x + (3D - 4C)\cos x + 4Ax + (4B - 4A) = x - \sin x$

$4A = 1$, $4D + 3C = -1$, $3D - 4C = 0$, $4B - 4A = 0$ $\Rightarrow$ $A = 1/4$, $B = 1/4$, $C = -3/25$, $D = -4/25$

$y_p = (Ax+B) + (C\sin x + D\cos x)$

$= \left[\frac{1}{4}x + \frac{1}{4}\right] + \left[\frac{-3}{25}\sin x - \frac{4}{25}\cos x\right]$

Thus, $y_p = (Ax+B) + (C\sin x + D\cos x)$

$y_p = \left(\frac{x}{4} + \frac{1}{4}\right) + \left(\frac{-3}{25}\sin x - \frac{4}{25}\cos x\right)$

General Solution:-

$y = y_c + y_p$

$y = C_1 e^{2x} + x C_2 e^{2x} + \left(\frac{x}{4} + \frac{1}{4}\right) + \left(\frac{-3}{25}\sin x - \frac{4}{25}\cos x\right)$

Part B

These questions are generally more challenging, often highlighting important subtleties. They require a deeper understanding of each concept and the interrelations between them.

3. Solve the initial-value problems

1. $y'' - 2y' + 5y = \sin x, y(0) = 1, y'(0) = 1$

Homogeneous solution:

$$r^2 - 2r + 5 = 0$$

$$r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

$$\alpha = 1$$
$$\beta = 2$$

$$y_H = e^{\alpha x}\left[C_1 \cos(\beta x) + C_2 \sin(\beta x)\right]$$

$$y_H = e^{x}\left[C_1 \cos(2x) + C_2 \sin(2x)\right]$$

Particular solution

$$y_p = A \sin x + B \cos x$$
$$y_p' = A \cos x - B \sin x$$
$$y_p'' = -A \sin x - B \cos x$$

Sub. back in main DE, we get:

$$(-A \sin x - B \cos x) - 2(A \cos x - B \sin x) + 5(A \sin x + B \cos x) = \sin x$$

$$(2B + 4A) \sin x + (-2A + 4B) \cos x = \sin x$$

$$\therefore \left.\begin{aligned} 2B + 4A &= 1 \\ -2A + 4B &= 0 \end{aligned}\right\} \quad A = \frac{1}{5}, \; B = \frac{1}{10}$$

Condition-1 $\longrightarrow y(0) = 1$

$$y = e^{x}\left[C_1 \cos(2x) + C_2 \sin(2x)\right] + \frac{1}{5} \sin x + \frac{1}{10} \cos x$$

$$y' = e^{x}\left[C_1 \cos(2x) + C_2 \sin(2x) - 2C_1 \sin(2x) + 2C_2 \cos(2x)\right] + \frac{1}{5} \cos x - \frac{1}{10} \sin x$$

$$1 = \left[C_1 \cos(0) + \cancel{0}\right] + \cancel{\frac{1}{5}(0)} + \frac{1}{10}(1)$$

$$1 = C_1 + \frac{1}{10} \longrightarrow \boxed{C_1 = \frac{9}{10}}$$

Condition-2 $\longrightarrow y'(0) = 1$

$$1 = \left[C_1 + \cancel{C_2 \sin(0)} - \cancel{2C_1(0)} + 2C_2 \cos(0)\right] + \frac{1}{5}(1) + \cancel{0}$$

$$1 = C_1 + 2C_2 + \frac{1}{5} \quad \left[\text{We know } C_1 = 9/10\right]$$

$$\therefore \boxed{C_2 = \frac{-1}{20}}$$

$$y = e^{x}\left[\frac{9}{10} \cos(2x) - \frac{1}{20} \sin(2x)\right] + \frac{1}{5} \sin x + \frac{1}{10} \cos x$$

2. $y'' + y' - 2y = x + \sin 2x, y(0) = 1, y'(0) = 0$

Homogeneous Solution : $r^2 + r - 2 = 0$

$$r^2 + 2r - r - 2 = 0$$

$$(r+2)(r-1) = 0$$

$$r_1 = -2$$

$$r_2 = 1$$

Thus $y_H = C_1 e^{-2x} + C_2 e^{x}$

Particular Solution :

$$y_p = Ax + B + C\sin(2x) + D\cos(2x)$$

$$y_p' = A + 2C\cos(2x) - 2D\sin(2x)$$

$$y_p'' = -4C\sin(2x) - 4D\cos(2x)$$

Substituting back in main DE, we get

$$-4C\sin(2x) - 4D\cos(2x) + A + 2C\cos(2x) - 2D\sin(2x) - 2[Ax + B + C\sin(2x) + D\cos(2x)] = x + \sin(2x)$$

$$(-6C - 2D)\sin(2x) + (-6D + 2C)\cos(2x) + (A - 2B) - 2Ax = x + \sin(2x)$$

$-2A = 1 \longrightarrow A = -1/2$

$A - 2B = 0 \longrightarrow B = -1/4$

$-6C - 2D = 0$, $-6D + 2C = 0$ $\Big\}$ $C = -3/20$, $D = -1/20$

Thus

$$y_p = -\frac{x}{2} - \frac{1}{4} - \frac{3}{20}\sin(2x) - \frac{1}{20}\cos(2x)$$

$$y = C_1 e^{-2x} + C_2 e^{x} - \frac{x}{2} - \frac{1}{4} - \frac{3}{20}\sin(2x) - \frac{1}{20}\cos(2x)$$

Condition-1 $\longrightarrow y(0) = 1$

$$1 = C_1 + C_2 - \cancel{0} - \frac{1}{4} - \cancel{\frac{3}{20}\sin(0)} - \frac{1}{20}\cos(0)$$

$$1 = C_1 + C_2 - \frac{1}{4} - \frac{1}{20}$$

$$C_1 + C_2 = 1 + \frac{1}{4} + \frac{1}{20} = \frac{20 + 5 + 1}{20} = \frac{26}{20} = \frac{13}{10}$$

Condition-2 $\longrightarrow y'(0) = 0$

$$y = C_1 e^{-2x} + C_2 e^{x} - \frac{x}{2} - \frac{1}{4} - \frac{3}{20}\sin(2x) - \frac{1}{20}\cos(2x)$$

$$y' = -2C_1 e^{-2x} + C_2 e^{x} - \frac{1}{2} - \frac{6}{20}\cos(2x) + \frac{1}{10}\sin(2x)$$

$$0 = -2C_1 + C_2 - \frac{1}{2} - \frac{6}{20}\cos(0) + \cancel{\frac{1}{10}\sin(0)}$$

$$0 = -2C_1 + C_2 - \frac{1}{2} - \frac{6}{20}$$

$$\frac{16}{20} = -2C_1 + C_2$$

Solving

$$\left.\begin{aligned} C_1 + C_2 &= 1.3 \\ -2C_1 + C_2 &= 0.8 \end{aligned}\right\} \quad \begin{aligned} C_1 &= 0.16 \\ C_2 &= 1.13 \end{aligned}$$

Thus,

$$y = (0.16)e^{-2x} + (1.13)e^{x} - \frac{x}{2} - \frac{1}{4} - \frac{3}{20}\sin(2x) - \frac{1}{20}\cos(2x)$$