Math 54 Worksheet

Second-Order Homogeneous Linear Equations

Part A

These questions test your knowledge of the core concepts and computations.

1. Find the general solution to the following Differential Equations

1. $y'' + y' - 6y = 0$ [Real and Distinct]

$r^2 + r - 6 = 0$

$(r+3)(r-2) = 0$

$r_1 = -3$

$r_2 = 2$

$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

$= C_1 e^{-3x} + C_2 e^{2x}$

2. $3\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$

$3r^2 + r - 1 = 0$

$r = \dfrac{-1 \pm \sqrt{1+12}}{6}$

$r = \dfrac{-1 \pm \sqrt{13}}{6}$

$r_1 = \dfrac{-1+\sqrt{13}}{6}$

$r_2 = \dfrac{-1-\sqrt{13}}{6}$

$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

$= C_1 e^{\left(\frac{-1+\sqrt{13}}{6}\right)x} + C_2 e^{\left(\frac{-1-\sqrt{13}}{6}\right)x}$

3. $4y'' + 12y' + 9y = 0$ [Repeated Real Roots]

$4r^2 + 12r + 9 = 0$

Determinant $\Delta = b^2 - 4ac$

$= 12^2 - 4(4)(9)$

$= 0$

$r_1 \,\&\, r_2 = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{-b}{2a}$

$r_1 = r_2 = \dfrac{-b}{2a} = \dfrac{-12}{8} = \dfrac{-3}{2}$

$y = C_1 e^{r_1 x} + C_2 x e^{r_2 x}$

$= C_1 e^{-\frac{3}{2}x} + C_2 e^{-\frac{3}{2}x}$

4. $y'' - 6y' + 13y = 0$ [Complex Roots]

$r^2 - 6r + 13 = 0$

$r = \dfrac{6 \pm \sqrt{36 - 13(4)}}{2} = \dfrac{6 \pm \sqrt{-16}}{2}$

$r_1 = 3 + 2i$

$r_2 = 3 - 2i$

$y = e^{ax}\left[C_1 \operatorname{Cos}(Bx) + C_2 \operatorname{Sin}(Bx)\right]$

$= e^{3x}\left[C_1 \operatorname{Cos}(2x) + C_2 \operatorname{Sin}(2x)\right]$

Part B

These questions are generally more challenging, often highlighting important subtleties. They require a deeper understanding of each concept and the interrelations between them.

2. Solve the following initial-value problems, if possible

1. $y'' + y' - 6y = 0, y(0) = 1, y'(0) = 0$

$y = C_1 e^{-3x} + C_2 e^{2x}$ [Q1a]

$y(0) = 1$

$1 = C_1 e^{-0} + C_2 e^{0}$

$\boxed{C_1 + C_2 = 1}$

$y'(0) = 0$

$y = C_1 e^{-3x} + C_2 e^{2x}$

$y' = -3C_1 e^{-3x} + 2C_2 e^{2x}$

$\therefore y'(0) = 0 = -3C_1 + 2C_2$

$\boxed{-3C_1 + C_2 = 0}$

$C_1 = 2/5$

$C_2 = 3/5$

$\therefore y = \frac{2}{5} e^{-3x} + \frac{3}{5} e^{2x}$

2. $y'' + y = 0, y(0) = 2, y'(0) = 3$

$r^2 + 1 = 0$

$r = \pm\sqrt{-1}$

$= \pm i$

$r_1 = i$

$r_2 = -i$

$y = C_1 \cos x + C_2 \sin x$

$y' = -C_1 \sin x + C_2 \cos x$

$y(0) = 2$

$2 = C_1 + 0$

$\boxed{C_1 = 2}$

$y'(0) = 3$

$3 = -0 + C_2$

$\boxed{C_2 = 3}$

$\therefore \boxed{y = 2\cos x + 3\sin x}$

3. $y'' + 4y' + 20y = 0$, $y(0) = 1$, $y(\pi) = 2$

$r^2 + 4r + 20 = 0$

$r = \frac{-4 \pm 8i}{2}$

$r_1 = \frac{-4+8i}{2} = -2+4i$

$r_2 = \frac{-4-8i}{2} = -2-4i$

Thus

$y(x) = e^{\alpha x}[C_1 \cos(\beta x) + C_2 \sin(\beta x)]$

$y = e^{-2x}[C_1 \cos(4x) + C_2 \sin(4x)]$

$y(0) = 1$

$1 = e^{0}[C_1 \cos(0) + C_2 \sin(0)]$

$\boxed{C_1 = 1}$

$y(\pi) = 2$

$2 = e^{-2\pi}[C_1 \cos(4x) + C_2 \sin(4x)]$

$2 = e^{-2\pi}[1(1) + C_2(0)]$

$2 = e^{-2\pi}$

Contradictory

Thus Boundary Value Problem has

No Solution