HANDOUT 19

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Singular Value Decomposition

Exercise: 1. For each of the following matrices, find all singular values.

2. Find the SVD of each of the matrices.

(1) $\begin{pmatrix} 0 & 6 \\ 2 & 0 \end{pmatrix}$

(2) $\begin{pmatrix} 4 & 6 \\ 0 & 4 \end{pmatrix}$

(3) $\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 6 & 0 \end{pmatrix}$

(4) $\begin{pmatrix} 3 & 0 & 1 \\ 0 & 5 & 0 \end{pmatrix}$

(1) $A = \begin{pmatrix} 0 & 6 \\ 2 & 0 \end{pmatrix}$ $\quad A^TA = \begin{pmatrix} 0 & 2 \\ 6 & 0 \end{pmatrix}\begin{pmatrix} 0 & 6 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 36 \end{pmatrix}$

$\lambda_1 = 36 \quad \lambda_2 = 4 \quad \Rightarrow \sigma_1 = 6 \quad \sigma_2 = 2$

$(A^TA - \lambda_1 I)x = 0 \quad \begin{pmatrix} -32 & 0 \\ 0 & 0 \end{pmatrix} x = 0 \Rightarrow \{x = s\begin{pmatrix} 0 \\ 1 \end{pmatrix} : s \in \mathbb{R}\} \quad v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$(A^TA - \lambda_2 I)x = 0 \quad \begin{pmatrix} 0 & 0 \\ 0 & 32 \end{pmatrix} x = 0 \Rightarrow \{x = s\begin{pmatrix} 1 \\ 0 \end{pmatrix} : s \in \mathbb{R}\} \quad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{6}\begin{pmatrix} 0 & 6 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{2}\begin{pmatrix} 0 & 6 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$A = U\Sigma V^T \quad U = [u_1 \ u_2] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Sigma = \begin{pmatrix} 6 & \\ & 2 \end{pmatrix} \quad V = [v_1 \ v_2] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(2) $A = \begin{pmatrix} 4 & 6 \\ 0 & 4 \end{pmatrix} \quad A^TA = \begin{pmatrix} 4 & 0 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} 4 & 6 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 16 & 24 \\ 24 & 52 \end{pmatrix}$

$\det(A^TA - \lambda I) = \begin{vmatrix} 16-\lambda & 24 \\ 24 & 52-\lambda \end{vmatrix} = (16-\lambda)(52-\lambda) - 24^2 = \lambda^2 - 68\lambda + 256$

$= (\lambda - 64)(\lambda - 4)$

$\lambda_1 = 64 \quad \lambda_2 = 4 \quad \sigma_1 = 8 \quad \sigma_2 = 2$

$(A^TA - \lambda_1 I)x = 0 \quad \begin{pmatrix} -48 & 24 \\ 24 & -12 \end{pmatrix} x = 0 \quad \{x = s\begin{pmatrix} 1 \\ 2 \end{pmatrix} : s \in \mathbb{R}\} \quad v_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$(A^TA - \lambda_2 I)x = 0 \quad \begin{pmatrix} 12 & 24 \\ 24 & 48 \end{pmatrix} x = 0 \quad \{x = s\begin{pmatrix} -2 \\ 1 \end{pmatrix} : s \in \mathbb{R}\} \quad v_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{8}\begin{pmatrix} 4 & 6 \\ 0 & 4 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{8\sqrt{5}}\begin{pmatrix} 16 \\ 8 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{2}\begin{pmatrix}4 & 6\\ 0 & 4\end{pmatrix}\frac{1}{\sqrt5}\begin{pmatrix}-2\\ 1\end{pmatrix} = \frac{1}{2\sqrt5}\begin{pmatrix}-2\\ 4\end{pmatrix} = \frac{1}{\sqrt5}\begin{pmatrix}-1\\ 2\end{pmatrix}$

$A = U\Sigma V^T$ $\quad U = [u_1\ u_2] = \frac{1}{\sqrt5}\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}$ $\quad \Sigma = \begin{pmatrix}8 & 0\\ 0 & 2\end{pmatrix}$ $\quad V = \frac{1}{\sqrt5}\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}$

(3) $A = \begin{pmatrix}0 & 1\\ 0 & 1\\ 6 & 0\end{pmatrix}$ $\quad A^TA = \begin{pmatrix}0 & 0 & 6\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 1\\ 6 & 0\end{pmatrix} = \begin{pmatrix}36 & 0\\ 0 & 2\end{pmatrix}$

$\det(A^TA - \lambda I) = \begin{vmatrix}36-\lambda & 0\\ 0 & 2-\lambda\end{vmatrix} = 0$ $\quad (\lambda-36)(\lambda-2) = 0$ $\quad \lambda_1 = 36 \quad \lambda_2 = 2$

$\sigma_1 = \sqrt{\lambda_1} = 6 \qquad \sigma_2 = \sqrt{\lambda_2} = \sqrt2$

$(A^TA - \lambda_1 I)x = 0 \qquad \begin{pmatrix}0 & 0\\ 0 & -34\end{pmatrix}x = 0 \quad \left\{x = s\begin{pmatrix}1\\ 0\end{pmatrix} : s \in \mathbb{R}\right\} \quad v_1 = \begin{pmatrix}1\\ 0\end{pmatrix}$

$(A^TA - \lambda_2 I)x = 0 \qquad \begin{pmatrix}34 & 0\\ 0 & 0\end{pmatrix}x = 0 \quad \left\{x = s\begin{pmatrix}0\\ 1\end{pmatrix} : s \in \mathbb{R}\right\} \quad v_2 = \begin{pmatrix}0\\ 1\end{pmatrix}$

$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{6}\begin{pmatrix}0 & 1\\ 0 & 1\\ 6 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$

$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt2}\begin{pmatrix}0 & 1\\ 0 & 1\\ 6 & 0\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \frac{1}{\sqrt2}\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}$

for u_3. we know that $\text{span}\{u_1, u_2, u_3\} = \mathbb{R}^3$ since U is orthogonal matrix

we pick x_3 from $\mathbb{R}^3$ and x_3 is not a linear combination.

for example: $\quad x_3 = \begin{pmatrix}1\\ -1\\ 0\end{pmatrix}$

then $\quad x_3 - (x_3 \cdot u_1)u_1 - (x_3 \cdot u_2)u_2 = \begin{pmatrix}1\\ -1\\ 0\end{pmatrix}$ is orthorgonal to u_1, u_2

so $\quad u_3 = \dfrac{\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}}{\left\|\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}\right\|} = \frac{1}{\sqrt2}\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}$

$U = [u_1\ u_2\ u_3] = \begin{pmatrix}0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2}\\ 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2}\\ 1 & 0 & 0\end{pmatrix}$ $\quad \Sigma = \begin{pmatrix}6 & 0\\ 0 & \sqrt2\\ 0 & 0\end{pmatrix}$ $\quad V = [v_1\ v_2] = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

(4) $A = \begin{pmatrix}3 & 0 & 1\\ 0 & 5 & 0\end{pmatrix}$ $\quad A^TA = \begin{pmatrix}3 & 0\\ 0 & 5\\ 1 & 0\end{pmatrix}\begin{pmatrix}3 & 0 & 1\\ 0 & 5 & 0\end{pmatrix} = \begin{pmatrix}9 & 0 & 3\\ 0 & 25 & 0\\ 3 & 0 & 1\end{pmatrix}$

$\det(A^TA - \lambda I) = 0 \Rightarrow \lambda_1 = 25 \quad \lambda_2 = 10. \quad \lambda_3 = 0$

$\sigma_1 = \sqrt{\lambda_1} = 5 \qquad \sigma_2 = \sqrt{\lambda_2} = \sqrt{10} \qquad \Sigma = \begin{pmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\end{pmatrix} = \begin{pmatrix}5 & 0 & 0\\ 0 & \sqrt{10} & 0\end{pmatrix}$

$V = [v_1\ v_2\ v_3]$

$$(A^TA-\lambda_1 I)x=0 \qquad \begin{pmatrix} -16 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & -24 \end{pmatrix} x = 0 \qquad \left\{ x = s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : s \in \mathbb{R} \right\} \qquad v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$(A^TA-\lambda_2 I)x=0 \qquad \begin{pmatrix} -1 & 0 & 3 \\ 0 & 15 & 0 \\ 3 & 0 & -9 \end{pmatrix} x = 0 \qquad \left\{ x = s\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} : s \in \mathbb{R} \right\} \qquad v_2 = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

$$(A^TA-\lambda_3 I)x=0 \qquad \begin{pmatrix} 9 & 0 & 3 \\ 0 & 25 & 0 \\ 3 & 0 & 1 \end{pmatrix} x = 0 \qquad \left\{ x = s\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} : s \in \mathbb{R} \right\} \qquad v_3 = \frac{1}{\sqrt{10}}\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

$$U = [u_1, u_2]$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{5}\begin{pmatrix} 3 & 0 & 1 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 & 0 & 1 \\ 0 & 5 & 0 \end{pmatrix}\frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$U = [u_1 \ u_2] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{pmatrix}$$

$$V = [v_1 \ v_2 \ v_3] = \begin{pmatrix} 0 & \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}$$