HANDOUT 18

Exercise 1.

(1)

$\|v_1\|^2 = \int_{-1}^{1} 1^2\, dx = 2$, $\|v_1\| = \sqrt{2}$

$\|v_2\|^2 = \int_{-1}^{1} x^2\, dx = 2\int_0^1 x^2 dx = \frac{2}{3}$, $\|v_2\| = \sqrt{\frac{2}{3}}$

$\|v_3\|^2 = \int_{-1}^{1} (x^2)^2\, dx = 2\int_0^1 x^4 dx = \frac{2}{5}$, $\|v_3\| = \sqrt{\frac{2}{5}}$

- $v_1 \cdot v_2 = \int_{-1}^{1} 1 \cdot x\, dx = 0$

$v_2 \cdot v_3 = \int_{-1}^{1} x \cdot x^2\, dx = 0$

$v_3 \cdot v_1 = \int_{-1}^{1} x^2 \cdot 1\, dx = 2\int_0^1 x^2 dx = 2 \cdot \frac{1}{3} = \frac{2}{3}$

(2) $v_1, v_2, v_3 \xrightarrow{\text{Gram-Schmidt}} u_1, u_2, u_3$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}$$

$v_1 \perp v_2 \Rightarrow u_1 \perp v_2$

$\Rightarrow \text{proj}_{u_1} v_2 = 0$

$$u_2 = \frac{v_2 - \text{proj}_{u_1} v_2}{\|v_2 - \text{proj}_{u_1} v_2\|} = \frac{v_2}{\|v_2\|} = \sqrt{\frac{3}{2}}\, x$$

$$u_3 = \frac{v_3 - \text{proj}_{u_1} v_3 - \text{proj}_{u_2} v_3}{\|v_3 - \text{proj}_{u_1} v_3 - \text{proj}_{u_2} v_3\|}$$

$\text{proj}_{u_2} v_3 = 0$

$$\text{proj}_{u_1} v_3 = \text{proj}_{v_1} v_3 = \frac{v_3 \cdot v_1}{v_1 \cdot v_1} v_1 = \frac{\frac{2}{3}}{2} v_1 = \frac{1}{3} v_1$$

$$= \frac{v_3 - \frac{1}{3} v_1}{\|v_3 - \frac{1}{3} v_1\|}$$

$$= \sqrt{\frac{45}{8}}\left(v_3 - \frac{1}{3} v_1\right)$$

$$= \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

$$\|v_3 - \tfrac{1}{3} v_1\|^2$$

$$= (v_3 - \tfrac{1}{3} v_1) \cdot (v_3 - \tfrac{1}{3} v_1)$$

$$= v_3 \cdot v_3 - \tfrac{2}{3} v_3 \cdot v_1 + \tfrac{1}{9} v_1 \cdot v_1$$

$$= \frac{2}{5} - \frac{2}{3} \cdot \frac{2}{3} + \frac{1}{9} \cdot 2$$

$$= \frac{8}{45}$$

$$\{u_1, u_2, u_3\} = \left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)\right\}$$

Exercise 2.

(1) $(A^2)^T = (A \cdot A)^T = A^T \cdot A^T = (A^T)^2$ (Use $(AB)^T = B^T A^T$)

Since A is symmetric, $A^T = A$

hence $(A^2)^T = (A^T)^2 = A^2$

(2)

$A = PDP^T$ (Use $(ABC)^T = C^T B^T A^T$)

$\Rightarrow A^T = (PDP^T)^T = (P^T)^T D^T P^T = PD^T P^T = A$

(3)

$A = PDP^T$

$\Rightarrow A^2 = (PDP^T)(PDP^T) = PD(P^T P) \cdot DP^T = PD \cdot DP^T = PD^2 P^T$

Hence A^2 is orthogonally diagonalizable

Exercise 3.

(1)

(i) Characteristic equation.

$$\det\begin{pmatrix} 9-\lambda & -1 \\ -1 & 9-\lambda \end{pmatrix} = 0 \Leftrightarrow \lambda^2 - 18\lambda + 80 = 0 \Leftrightarrow (\lambda-10)(\lambda-8) = 0$$

$$\Rightarrow \lambda_1 = 10,\ \lambda_2 = 8$$

(ii) Solve $(A - \lambda_1 I)x = 0$

$$\Leftrightarrow \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Leftrightarrow x_1 + x_2 = 0$$

λ_1 - Eigenvector $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Solve $(A - \lambda_2)x = 0$

$$\Leftrightarrow \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Leftrightarrow x_1 - x_2 = 0$$

λ_2 - Eigenvector $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(iii) Apply Gram-Schmidt to v_1, v_2 to get u_1, u_2

$$u_1 = \frac{v_1}{\|v_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \text{proj}_{u_1} v_2 = \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = 0$$

$$u_2 = \frac{v_2 - \text{proj}_{u_1} v_2}{\|v_2 - \text{proj}_{u_1} v_2\|} = \frac{v_2}{\|v_2\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

(iv) $P = (u_1, u_2) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $D = \text{diag}(10, 8) = \begin{bmatrix} 10 & 0 \\ 0 & 8 \end{bmatrix}$

$$A = PDP^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(12).

Characteristic equation

$$\det \begin{pmatrix} 3-\lambda & -2 & 4 \\ -2 & 6-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix} = 0 \Leftrightarrow (3-\lambda) \begin{vmatrix} 6-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} + 2 \begin{vmatrix} -2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4 \begin{vmatrix} -2 & 6-\lambda \\ 4 & 2 \end{vmatrix} = 0$$

$$\Leftrightarrow (3-\lambda)\left((6-\lambda)(3-\lambda)-4\right) + 2(-2(3-\lambda)-8) + 4(-4-4(6-\lambda)) = 0$$

$$\Leftrightarrow \underbrace{\lambda^3 - 12\lambda^2 + 21\lambda + 98}_{f(\lambda)} = 0$$

$98 = 14 \times 7$
$= 2 \times 7 \times 7$

To solve the equation: Guess $\lambda = 7$ is a root ✓

Actually a double root (The derivative $f'(\lambda) = 3\lambda^2 - 24\lambda + 21$
$= 3(\lambda - 7)(\lambda - 1)$
$f'(7) = 0$).

$$\Rightarrow f(\lambda) = (\lambda - 7)^2 (\lambda + 2)$$

$\lambda_1 = 7, \lambda_2 = 7, \lambda_3 = -2.$

$\lambda_1 = \lambda_2$

Solve $(A-\lambda_1 I)x=0$

$$\Leftrightarrow \begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Row reduction

$$\begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ -2 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-2x_1 - x_2 + 2x_3 = 0$$

Basis for λ_1 eigen-space $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

Solve $(A-\lambda_3 I)x=0$

$$\Leftrightarrow \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Row reduction

Row reduction

$$\begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -1 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & -1 \\ 0 & -1 & 0 \\ 0 & 18 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\begin{cases} x_1 + x_3 = 0 \\ 2x_2 + x_3 = 0 \end{cases} \qquad \lambda_3\text{- eigenvector } V_3 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

(iii). Apply Gram Schmidt Process for V_1, V_2, V_3 to get u_1, u_2, u_3.

$$u_1 = \frac{V_1}{\|V_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \text{proj}_{u_1} V_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

$$u_2 = \frac{V_2 - \text{proj}_{u_1} V_2}{\|V_2 - \text{proj}_{u_1} V_2\|} \qquad V_2 - \text{proj}_{u_1} V_2 = \begin{bmatrix} -\frac{1}{2} \\ 2 \\ \frac{1}{2} \end{bmatrix}$$

$$\|V_2 - \text{proj}_{u_1} V_2\| = \frac{3}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{3} \begin{bmatrix} -\frac{1}{2} \\ 2 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3\sqrt{2}} \\ \frac{2\sqrt{2}}{3} \\ \frac{1}{3\sqrt{2}} \end{bmatrix}$$

$$V_3 \perp \text{span}\{V_1, V_2\} \Rightarrow V_3 \perp \text{span}\{u_1, u_2\} \Rightarrow \text{proj}_{u_1} V_3 = \text{proj}_{u_2} V_3 = 0$$

$$u_3 = \frac{V_3}{\|V_3\|} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

(iv) $P = (u_1, u_2, u_3) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & -\frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \end{bmatrix}$

$D = \begin{bmatrix} 7 & & \\ & 7 & \\ & & -2 \end{bmatrix}$

$A = PDP^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & -\frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 7 & & \\ & 7 & \\ & & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} & \frac{2\sqrt{2}}{3} & \frac{1}{3\sqrt{2}} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$