HANDOUT 18

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Inner product: $\forall u, v \in V$, where V is a vector space,

(1) $\langle u, v\rangle = \langle v, u\rangle$
(2) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$
(3) $\langle cu, v\rangle = c\langle u, v\rangle, \quad \forall c \in \mathbb{R}$
(4) $\langle u, u\rangle = 0$ if and only if $u = 0$

Exercise 1. Put the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$ on the space of continuous functions on $[-1, 1]$.

(1) Let $v_1 = 1, v_2 = x, v_3 = x^2$. Find lengths of v_1, v_2, v_3 and inner products $v_1 \cdot v_2, v_2 \cdot v_3, v_3 \cdot v_1$.
(2) Apply Gram-Schmidt to the aforementioned v_1, v_2, v_3.

Symmetric matrix: $A = A^\top$

Orthogonally diagonalizable: there exists orthogonal matrix P and a diagonal matrix D such that $A = PDP^\top$.

Exercise 2. Let A be an $n \times n$ matrix:

(1) Show that if A is symmetric, then so is A^2.
(2) Show that if A is orthogonally diagonalizable, then it is symmetric.
(3) Show that if A is orthogonally diagonalizable, then so is A^2.

How to orthorgonally diagonalize a symmetric matrix A:

(1) Write down characteristic equation, solve it and obtain eigenvalues λ_1, λ_2, ...,c_n
(2) Solve $(A - \lambda_i I)x = 0$ and obtain eigenvector v_i,
(3) Normalize v_i to u_i (Might use Gram-Schmidt process if geometric multiplicity > 1).
(4) $A = PDP^\top$, where $P = (u_1, u_2, \cdots, u_n)$ and $D = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

Exercise 3. Orthogonally diagonalize the matrix:

(1) $\begin{pmatrix} 9 & -1 \\ -1 & 9 \end{pmatrix}$ (2) $\begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}$ (Textbook section 7.1 example 3)