MIDTERM REVIEW 2

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Make sure you know how to compute:

(1) determinant of square matrix.
(2) inner product, norm, distance between two points in $\mathbb{R}^n$.
(3) the Gram-Schmidt process

Make sure you know the definition of:

(1) Vector space. The properties a set should satisfy in order to be a vectors space.
(2) Subspace. The properties a **subset** should satisfy in order to be a subspace.
(3) Linear transformation. The properties a transformation should satisfy in order to be a linear transformation.
(4) Kernel/Null space, column space, row space, basis.
    - rank $A = \dim \text{Col}(A) = \dim \text{Row}(A)$
    - If A is of size $m \times n$, $n = \dim \text{Nul}(A) + \dim \text{Col}(A)$, $m = \dim \text{Nul}(A^\top) + \dim \text{Row}(A)$
(5) Eigenvalues, eigenvectors and eigenspaces of a matrix.
(6) Eigenvalues, eigenvectors and eigenspaces of a linear transformation.

Make sure you know how to solve the following problem:

(1) Linear independent:
    - How to determine if the sets of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is Linear independent.
        - Solve equation $(\mathbf{v}_1\mathbf{v}_2 \cdots \mathbf{v}_n)\mathbf{x} = 0$. If only trivial solution, linearly independent, otherwise, not.
(2) Eigenvalue and eigenvector:
    - How to find all the eigenvalues and eigenvectors of matrix A.
        - Solve the characteristic equation, $\det(A - \lambda I) = 0$. The roots $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues. For eigenvalue λ_i, solve equation $(A - \lambda_i)\mathbf{x} = 0$. The solution set is the corresponding eigenspace. We may choose the vectors in a basis for the solution set as eigenvectors.
    - How to compute the algebraic and geometric multiplicity of eigenvalues of matrix A.
        - The algebraic multiplicity of eigenvalue λ is its multiplicity as a root of the characteristic equation. The geometric multiplicity of eigenvalue λ is the dimension of the null space of $A - \lambda I$.
(3) Diagonalize matrix:
    - $n \times n$ matrix A is diagonalizable $\Longleftrightarrow$ it has n linear independent eigenvectors $\Longleftrightarrow$ For any eigenvalue of matrix A, λ, the algebraic multiplicity of λ equals its geometric multiplicity.
    - How to determine if a matrix A is diagonalizable.
        - Calculate the algebraic and algebraic multiplicity of the eigenvalues of matrix A. If alg. multi. = geo. multi. for all eigenvalues, then A is diagonalizable, otherwise, not.
    - How to diagonalize a matrix.
        - Find all the eigenvalues of A,$\lambda_1, \lambda_2, \cdots, \lambda_n$, as well as the corresponding eigenvectors, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$. Then let $P = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n)$

$$P^{-1}AP = diag(\lambda_1, \lambda_2, \cdots, \lambda_n).$$

(4) Similar
- How to determine if matrix A and B are similar.
  - If both A and B are diagonalizable, we only need to check whether A and B have the same set of eigenvalues. If same, $A \sim B$, otherwise, not
  - If only one of two matrices is diagonalizble, then A is not similar to B.
  - The rest case is too complicated and needs the knowledge of Jordan form. I don't think it will be tested in midterm 2.

(5) Orthogonal:
- How to determine if the sets of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthogonal.
  - Calculate $\mathbf{v}_1 \cdot \mathbf{v}_2$, $\mathbf{v}_1 \cdot \mathbf{v}_3$, $\mathbf{v}_2 \cdot \mathbf{v}_3$. If all zeros, orthogonal, otherwise, not orthogonal.

(6) Orthonormal:
- How to determine if the sets of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthonormal.
  - Step 1: Calculate $\mathbf{v}_1 \cdot \mathbf{v}_2$, $\mathbf{v}_1 \cdot \mathbf{v}_3$, $\mathbf{v}_2 \cdot \mathbf{v}_3$. If all zeros, orthogonal, otherwise, not.
  - Step 2: calculate $\|\mathbf{v}_1\|$, $\|\mathbf{v}_2\|$ and $\|\mathbf{v}_3\|$. If all 1s, orthonormal, otherwise, not.

(7) Least squares problem:
- How to solve the least square problem for A and b
  - Solve $A^\top Ax = A^\top b$.

The last part is about coordinate system:

(1) Suppose V is a vector space with a basis $\mathcal{B} = \{b_1, b_2, \cdots, b_n\}$. v is a vector of V.
- How to compute the $\mathcal{B}$-coordinate of v
  - Write v as a linear combination of $b_1, b_2, \cdots, b_n$. If $V = \mathbb{R}^n$, then we only need to solve matrix equation

$$(b_1 \quad b_2 \quad \cdots \quad b_n)x = v.$$

Its solution is $[x]_\mathcal{B}$.

(2) Suppose V has another basis $\mathcal{C} = \{c_1, c_2, \cdots, c_n\}$.
- How to compute the matrix changing basis from $\mathcal{B}$ to $\mathcal{C}$
  - Calculate the C-coordinate of $b_1, b_2, \cdots, b_n$

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \Big[[b_1]_\mathcal{C} \quad [b_2]_\mathcal{C} \quad \cdots \quad [b_n]_\mathcal{C}\Big], \qquad [x]_\mathcal{C} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[x]_\mathcal{B}.$$

(3) Suppose $T : V \to V$ is a linear transformation.
- How to compute the $\mathcal{B}$-matrix of T
  - Calculate the $\mathcal{B}$-coordinate of $T(b_1), T(b_2), \cdots, T(b_n)$

$$T_\mathcal{B} = \Big[[T(b_1)]_\mathcal{B} \quad [T(b_2)]_\mathcal{B} \quad \cdots \quad [T(b_2)]_\mathcal{B}\Big], \quad [T(v)]_\mathcal{B} = T_\mathcal{B}[v]_\mathcal{B}.$$

- What is the relation between the $\mathcal{B}$-matrix of T, $T_\mathcal{B}$ and the $\mathcal{C}$-matrix of T, $T_\mathcal{C}$

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P^{-1}} T_\mathcal{C} \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = T_\mathcal{B}.$$