for λ
algebraic multiplicity: the multiplicity of root λ in characteristic eq.
example: $(\lambda-2)^3=0$ alg. multiplicity of $\lambda=2$ is 3.
geometric multiplicity: the dimension of $Nul(A-\lambda I)$

HANDOUT 13

JIASU WANG

- Diagonalizable, eigenvector basis, (algebraic/geometric) multiplicity, similar

  Matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

  Exercise1: What are the eigenvalues of A and their algebraic multiplicities if A is similar to the following diagonal matrix?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

eigenvalues: $\lambda=1, 3, 2$
multiplicity for $\lambda=1$: 2
multiplicity for $\lambda=3$: 1
multiplicity for $\lambda=2$: 1

- How to diagonalize matrix

  Exercise2: Is the following matrix diagonalizable? If so, diagonalize it.

$$A=\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$\det(A-\lambda I)=(2-\lambda)(3-\lambda)^2$
characteristic eq.
$(2-\lambda)(3-\lambda)^2=0$
roots: $\lambda=2, \lambda=3$
① $\lambda=2$. $(A-2I)x=0$
solution set: $\left\{x=\begin{pmatrix}1\\-1\\-1\end{pmatrix}s : s\in\mathbb{R}\right\}$

② $\lambda=3$ $(A-3I)x=0$
solution set
$\left\{x=\begin{pmatrix}0\\1\\0\end{pmatrix}s+\begin{pmatrix}0\\0\\1\end{pmatrix}t : s,t\in\mathbb{R}\right\}$
3 eigenvectors: $v_1=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$, $v_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $v_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$
$\Rightarrow$ by definition. A is diagonalizable.
Let $P=(v_1\ v_2\ v_3)$, $P^{-1}AP=\begin{pmatrix}2&0&0\\0&3&0\\0&0&3\end{pmatrix}$

  Exercise3: Determine if the following matrices are diagonalizable. Determine also if they are similar.

  (1) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.

  (2) $\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ and $\begin{bmatrix} -1 & 3 \\ -4 & 6 \end{bmatrix}$.

(1) $A=\begin{pmatrix}2&1\\0&2\end{pmatrix}$ $\det(A-\lambda I)=(2-\lambda)^2$
characteristic eq. $(2-\lambda)^2=0$
$\lambda=2$. $(A-2I)x=0$
Sol. set $\left\{x=\begin{pmatrix}1\\0\end{pmatrix}s : s\in\mathbb{R}\right\}$
$\Rightarrow$ eigenvector $\begin{pmatrix}1\\0\end{pmatrix}$

$B=\begin{pmatrix}2&3\\0&2\end{pmatrix}$ $\det(B-\lambda I)=(2-\lambda)^2$
characteristic eq. $(2-\lambda)^2$
$\lambda=2$ $(A-2I)x=0$
Sol. set $\left\{x=\begin{pmatrix}1\\0\end{pmatrix}s : s\in\mathbb{R}\right\}$
$\Rightarrow$ eigenvector $\begin{pmatrix}1\\0\end{pmatrix}$

the number of eigenvectors = 1 < 2

$\Rightarrow$ A not diagonalizable.

the number of eigenvectors = 1 < 2

$\Rightarrow$ B not diagonalizable.

However A and B are similar because the alg. and geo. multiplicity of $\lambda=2$ for A equals that of $\lambda=2$ for B.

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$\Rightarrow PBP^{-1} = A$ where $P = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

matrix A is similar to B iff both the algebraic and geometric multiplicity of eigenvalues of A equals that of B

example: A

λ	alg.	geo.
3	2	1
2	1	1

B :

λ	alg.	geo
3	2	1
2	1	1

$\Rightarrow A \sim B$

in Exercise 3 (1)

$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

λ	alg.	geo.
2	2	1

$B = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$

λ	alg.	geo.
2	2	1

$\Rightarrow A \sim B$. although they are not diagonalizable.

(2) $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ $\det(A-\lambda I) = (1-\lambda)(4-\lambda)+2 = \lambda^2 - 5\lambda + 6$

characteristic eq. $\lambda^2 - 5\lambda + 6 = 0$

$(\lambda^2 - 5\lambda + 6) = (\lambda-2)(\lambda-3) \Rightarrow \lambda = 2$. or 3

$\lambda = 2$. $(A-2I)x = 0$ $\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}x = 0$

sol. set $\left\{ x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} s : s \in \mathbb{R} \right\}$

$\lambda = 3$ $(A-3I)x = 0$ $\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}x = 0$

sol. set $\left\{ x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t : t \in \mathbb{R} \right\}$

$\Rightarrow$ A diagonalizable

$B = \begin{pmatrix} -1 & 3 \\ -4 & 6 \end{pmatrix}$ $\det(B-\lambda I) = (-1-\lambda)(6-\lambda)+12 = \lambda^2 - 5\lambda + 6$

characteristic eq. $\lambda^2 - 5\lambda + 6 = 0$

$\lambda = 2$ or 3

$\lambda = 2$ $(B-2I)x = 0$ $\begin{pmatrix} -3 & 3 \\ -4 & 4 \end{pmatrix}x = 0$

sol. set $\left\{ x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} s : s \in \mathbb{R} \right\}$

$\lambda = 3$ $(B-3I)x = 0$ $\begin{pmatrix} -4 & 3 \\ -4 & 3 \end{pmatrix}x = 0$

sol. set $\left\{ x = \begin{pmatrix} 3 \\ 4 \end{pmatrix} t : t \in \mathbb{R} \right\}$

$\Rightarrow$ B diagonalizable.

A

λ	alg.	geo.
2	1	1
3	1	1

$A \sim B$

B

λ	alg.	geo
2	1	1
3	1	1

- Complex eigenvalue: The roots of characteristic equation can be complex.

  Exercise4: Find the eigenvalues of the following matrix, ~~as well as the corresponding matrix~~

$$A = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$$

$\det(A-\lambda I) = (1-\lambda)(5-\lambda)+8 = \lambda^2 - 6\lambda + 13$

characteristic eq: $\lambda^2 - 6\lambda + 13 = 0$

$\lambda^2 - 6\lambda + 9 + 4 = 0$

$(\lambda-3)^2 + 4 = 0$

$(\lambda-3)^2 = -4$

$\lambda - 3 = \pm\sqrt{-4}$

$\lambda = 3 \pm 2i$

(or $\lambda = 3+2i,\ 3-2i$)

$\sqrt{-4} = \sqrt{4}\cdot\sqrt{-1} = 2i$

- Eigenvalue and eigenvector of linear transformation.

  Exercise5: Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that $T(x) = Ax$. Consider vector $v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Find $T(v)$. Is v an eigenvector of A? If so, what is the associated eigenvalue?

$T(v) = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 4v$

$\Rightarrow$ v is an eigenvector of T

the associated eigenvalue is 4

- Matrix of a linear transformation Given a linear transformation $T : V \to V$ and vector space V with basis $\mathcal{B} = \{b_1, b_2, \cdots, b_n\}$. For any $x \in V$, if $x = r_1b_1 + r_2b_2 + \cdots + r_nb_n$, then

rank (T) = rank of matrix $T_{\mathcal{B}}$

$$[x]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}, \quad [T(x)]]_{\mathcal{B}} = \underbrace{\Big[[T(b_1)]_{\mathcal{B}} \quad [T(b_2)]_{\mathcal{B}} \quad \cdots \quad [T(b_n)]_{\mathcal{B}}\Big]}_{T_{\mathcal{B}}}[x]_{\mathcal{B}}.$$

Exercise6: (Geometric interpretation of similar matrices) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(e_1) = e_1 - e_2, T(e_2) = e_1 + e_2$.

(1) Find the matrix T_B of T under the basis $B = \{e_1, e_2\}$.
(2) Find the matrix T_C of T under the basis $C = \{e_1, e_1 + e_2\}$.
(3) Find the change of coordinate matrix from B to C, i.e., a matrix A such that the coordinate of a vector under basis B is the coordinate of the vector under basis B, left multiplied by the matrix A.
(4) Show that $\cancel{T_B = AT_CA^{-1}}$. $T_C = AT_BA^{-1}$

$B = \{e_1, e_2\}$

(1) $T_B = \big[\,[T(e_1)]_B \quad [T(e_2)]_B\,\big]$

$T(e_1) = e_1 - e_2 \Rightarrow [T(e_1)]_B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$T(e_2) = e_1 + e_2 \Rightarrow [T(e_2)]_B = \begin{pmatrix} 1 \\ \ \end{pmatrix}$

$\Bigg\} \Rightarrow T_B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

(2) $T_C = \left[\, [T(e_1)]_C \quad [T(e_1+e_2)]_C \,\right] \quad C = \{e_1, e_1+e_2\}$

$T(e_1) = e_1 - e_2 = 2e_1 - (e_1+e_2) \Rightarrow [T(e_1)]_C = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$T(e_1+e_2) = T(e_1) + T(e_2) = e_1 - e_2 + e_1 + e_2 = 2e_1 \Rightarrow [T(e_1+e_2)]_C = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

in conclusion. $T_C = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$

(3) $A = P_{B\to C} = \left[\, [e_1]_C \quad [e_2]_C \,\right]$

$[e_1]_C = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad e_2 = -e_1 + (e_1+e_2) \qquad [e_2]_C = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$A = P_{B\to C} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

(4) $A^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$A T_B A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 \\ -1 & 0 \end{pmatrix}$$

$\Rightarrow T_C = A T_B A^{-1}$

Exercise7: $Ab_1 = \begin{pmatrix} -2 \\ -16 \\ 8 \end{pmatrix}$ $Ab_2 = \begin{pmatrix} -1 \\ -7 \\ 4 \end{pmatrix}$ $Ab_3 = \begin{pmatrix} 12 \\ 15 \\ 3 \end{pmatrix}$

- Linear transformation on $\mathcal{RR}^n$ and matrix similarity

Exercise7: Find the $\mathcal{B}$-matrix for the transformation $x \to Ax$ when $\mathcal{B} = \{b_1, b_2, b_3\}$

$$A = \begin{bmatrix} -14 & 4 & -14 \\ -33 & 9 & -31 \\ 11 & -4 & 11 \end{bmatrix}, \quad b_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad b_3 = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$$

solve $(b_1\ b_2\ b_3)X = Ab_1$ $X = \begin{pmatrix} 20 \\ -12 \\ -6 \end{pmatrix}$ $[Ab_1]_{\mathcal{B}} = \begin{pmatrix} 20 \\ -12 \\ -6 \end{pmatrix}$ $\Rightarrow T_{\mathcal{B}} = \begin{pmatrix} 20 & 9 & 6 \\ -12 & -5 & -9 \\ -6 & -3 & -9 \end{pmatrix}$

solve $(b_1\ b_2\ b_3)X = Ab_2$ $X = \begin{pmatrix} 9 \\ -5 \\ -3 \end{pmatrix}$ $[Ab_2]_{\mathcal{B}} = \begin{pmatrix} 9 \\ -5 \\ -3 \end{pmatrix}$

simple method: calculate $P^{-1}AP$ where $P = (b_1\ b_2\ b_3)$

solve $(b_1\ b_2\ b_3)X = Ab_3$ $X = \begin{pmatrix} 6 \\ -9 \\ -9 \end{pmatrix}$ $[Ab_3]_{\mathcal{B}} = \begin{pmatrix} 6 \\ -9 \\ -9 \end{pmatrix}$

- Some problems about eigenvalues

Exercise8:Let $T : V \to V$ be a linear transformation with $\dim V = n$.

(1) Show that if T is not onto, then it has an eigenvector.

(2) Show that $\lambda \in \mathbb{R}$ is an eigenvalue for T if and only if $\operatorname{rank}(T - \lambda I) < n$.

(1) pf: Let $\mathcal{B} = \{b_1, b_2, \cdots, b_n\}$ denote a basis of V.

$\forall v \in V$. x has $\mathcal{B}$-coordinate $[v]_{\mathcal{B}}$.

$T: V \to V$ has $\mathcal{B}$ matrix. $T_{\mathcal{B}}$ i.e.

$[T(v)]_{\mathcal{B}} = T_{\mathcal{B}}[v]_{\mathcal{B}}$ here $T_{\mathcal{B}} = \Big[[T(b_1)]_{\mathcal{B}}\ [T(b_2)]_{\mathcal{B}} \cdots [T(b_n)]_{\mathcal{B}} \Big]$

since T is not onto. there exists $y \in V$ such that for any $v \in V$. $T(v) \neq y$

it implies there exsits $b = [y]_{\mathcal{B}} \in \mathbb{R}^n$ s.t $T_{\mathcal{B}}X = b$ has no sol.

$\Rightarrow \dim(\mathrm{Col}(T_{\mathcal{B}})) < n$

since $\dim(\mathrm{Col}(T_{\mathcal{B}})) + \dim(\mathrm{Nul}(T_{\mathcal{B}})) = n$

so $\dim(\mathrm{Nul}(T_{\mathcal{B}})) \geq 1$ There exists x_0 s.t $T_{\mathcal{B}}x_0 = 0$

let $v = (b_1\ b_2 \cdots b_n)x_0$ $[T(v)]_{\mathcal{B}} = T_{\mathcal{B}}x_0 = 0 \Rightarrow T(v) = 0$

$T(v) = 0 \cdot v$ $\Rightarrow$ v is an eigenvector.

(2) λ is an eigenvalue of T

$\Rightarrow$ there exists $v \neq 0$ such that $T(v) = \lambda v$

so $0 = T(v) - \lambda v = (T - \lambda I)(v)$

[cut off] transformation $T - \lambda I$ as S

denote the linear transformation [illegible]

we have $S(v) = 0$

Let $\mathcal{B} = \{b_1, b_2, \cdots, b_n\}$ denote a basis of V.

then we have $\mathcal{B}$-matrix for S, say $S_{\mathcal{B}}$.

$S(v) = 0$ means: $S_{\mathcal{B}} [v]_{\mathcal{B}} = 0$

$$\Rightarrow \dim(Nul(S_{\mathcal{B}})) > 0$$

Then, $\text{rank } S = \text{rank } S_{\mathcal{B}} = \dim Col(S_{\mathcal{B}})$

$$= n - \dim Nul(S_{\mathcal{B}}) < n.$$