HANDOUT 13

JIASU WANG

- Diagonalizable, eigenvector basis, (algebraic/geometric) multiplicity, similar

  Matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

  Exercise1: What are the eigenvalues of A and their algebraic multiplicities if A is similar to the following diagonal matrix?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

- How to diagonalize matrix

  Exercise2: Is the following matrix diagonalizable? If so, diagonalize it.

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

Exercise3: Determine if the following matrices are diagonalizable. Determine also if they are similar.

(1) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.

(2) $\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ and $\begin{bmatrix} -1 & 3 \\ -4 & 6 \end{bmatrix}$.

- Complex eigenvalue: The roots of characteristic equation can be complex.

  Exercise4: Find the eigenvalues of the following matrix, as well as the corresponding matrix

$$\begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$$

- Eigenvalue and eigenvector of linear transformation.

  Exercise5: Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that $T(x) = Ax$. Consider vector $v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Find $T(v)$. Is v an eigenvector of A? If so, what is the associated eigenvalue?

- Matrix of a linear transformation Given a linear transformation $T : V \to V$ and vector space V with basis $\mathcal{B} = \{b_1, b_2, \cdots, b_n\}$. For any $x \in V$, if $x = r_1 b_1 + r_2 b_2 + \cdots + r_n b_n$, then

$$[x]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}, \quad [T(x)]]_{\mathcal{B}} = \Big[[T(b_1)]_{\mathcal{B}} \quad [T(b_2)]_{\mathcal{B}} \quad \cdots \quad [T(b_n)]_{\mathcal{B}}\Big][x]_{\mathcal{B}}.$$

Exercise6: (Geometric interpretation of similar matrices) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(e_1) = e_1 - e_2, T(e_2) = e_1 + e_2$.

(1) Find the matrix T_B of T under the basis $B = \{e_1, e_2\}$.
(2) Find the matrix T_C of T under the basis $C = \{e_1, e_1 + e_2\}$.
(3) Find the change of coordinate matrix from B to C, i.e., a matrix A such that the coordinate of a vector under basis B is the coordinate of the vector under basis B, left multiplied by the matrix A.
(4) Show that $T_B = AT_C A^{-1}$.

- Linear transformation on $\mathcal{R}\mathcal{R}^n$ and matrix similarity

  Exercise7: Find the $\mathcal{B}$-matrix for the transformation $x \to Ax$ when $\mathcal{B} = \{b_1, b_2, b_3\}$

$$A = \begin{bmatrix} -14 & 4 & -14 \\ -33 & 9 & -31 \\ 11 & -4 & 11 \end{bmatrix}, \quad b_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad b_3 = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$$

- Some problems about eigenvalues

  Exercise8:Let $T : V \to V$ be a linear transformation with $\dim V = n$.

(1) Show that if T is not onto, then it has an eigenvector.
(2) Show that $\lambda \in \mathbb{R}$ is an eigenvalue for T if and only if $\operatorname{rank}(T - \lambda I) < n$.