HANDOUT 12

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Given a matrix A of size $n \times n$, the following statements are equivalent:

- $\det(A) \neq 0$,
- A is invertible,
- the columns of A are linear independent,
- the rows of A are linear independent,
- rank(A) = n,
- $Col(A) = \mathbb{R}^n$,
- $Row(A) = \mathbb{R}^n$,
- $Ax = b$ has unique solution for any b.

The following statements are equivalent:

- $\det(A) = 0$,
- A is not invertible,
- the columns of A are linear dependent,
- the rows of A are linear dependent,
- rank$(A) < n$,
- $Ax = b$ doesn't have unique solution for some b.

Note that when $\det(A) = 0$, $Ax = b$ can either have infinite solutions or no solution. It depends on the choice of b.

- Cramer's Rule

  Exercise: Use Cramer's rule to compute the solutions of the systems

$$\begin{cases} -5x_1 + 2x_2 = 9 \\ 3x_1 - x_2 = -4 \end{cases} \qquad \text{Sol: } \begin{cases} x_1 = 1 \\ x_2 = 7 \end{cases}$$

- Inverse formula

  Exercise: Compute the adjugate of the given matrix, and give the inverse of the matrix.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 3 & 0 & 6 \end{bmatrix} \qquad \text{Sol: } \frac{1}{3}\begin{bmatrix} -12 & -6 & 5 \\ -3 & 0 & 1 \\ 6 & 3 & 2 \end{bmatrix}$$

- Determinants as area or volume

  Exercise: Find the area of the parallelogram whose vertices are listed:

$$(-2, 0), \quad (0, 3), \quad (1, 3), \quad (-1, 0),. \text{ Sol: area} = 3.$$

- Characteristic equation, eigenvalue, eigenvector, eigenspace = null space of $(A - \lambda I)$

  λ is an eigenvalue of $A \Longrightarrow (A - \lambda I)x = 0$ has non trivial solutions.

  v is an eigenbalue of $A \Longrightarrow Av = \lambda v$ for some real number λ.

  Exercise: 1. Is $\lambda = 2$ is an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Yes

  Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvalue of $\begin{bmatrix} 2 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}$? Yes

2. Given matrix $\begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$, find a basis for the eigenspace corresponding to eigenvalue $\lambda = 1$. Basis: $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$

3. For each of the following matrices, find (i) its the characteristic equation, (ii) all of its eigenvalues, and (iii) a basis for each of its eigenspaces.

(1) $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$. $det(A - \lambda I) = (4 - \lambda)(1 - \lambda) = 0$, eigenvalues: $\lambda = 1$ or 4.

(2) $\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$. $det(A - \lambda I) = \lambda^2 - 2\lambda - 7 = 0$, eigenvalues: $\lambda = 1 \pm 2\sqrt{2}$.

(3) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. $det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 3\lambda = 0$, eigenvalues: $\lambda = 0, 1, 3$.