HANDOUT 10

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1. **Linear transformations**: A map between vector spaces is a linear transformation if it preserves addition and scalar multiplication.

Exercise: Determine if the following maps are linear transformations.

(1) The map $\mathbb{R}^m \to \mathbb{R}^n$ given by left multiplication of a matrix, that is, $T : x \mapsto Ax$.
 Sol: Yes
(2) $D = d/dx$ for real polynomials in x. (Note: The set of all real polynomials in variable x forms a vector space)
 Sol: Yes. $\mathbb{P} := \{$**all real polynomials in** $x\}$ is a vector space. D maps the vector in $\mathbb{P}$ to a vector in $\mathbb{P}$. D preserves addition and scalar multiplication.
(3) $D = d/dx$ for $\{(ax^2 + bx + c)e^x : a, b, c \in \mathbb{R}\}$ and $\{a \sin xe^x + b \cos xe^x : a, b, c \in \mathbb{R}\}$.
 Sol: Yes. $\mathbb{V} := \{(ax^2 + bx + c)e^x : a, b, c \in \mathbb{R}\}$ is a vector space. D maps the vector in $\mathbb{V}$ to a vector in $\mathbb{V}$. D preserves addition and scalar multiplication.
 Sol: Yes. $\mathbb{W} := \{a \sin xe^x + b \cos xe^x : a, b, c \in \mathbb{R}\}$ is a vector space. D maps the vector in $\mathbb{W}$ to a vector in $\mathbb{W}$. D preserves addition and scalar multiplication.

2. **Column space, Row space, Null spcae**:

- The pivot columns of a matrix A form a basis for $ColA$.
- The pivot rows of a matrix A form a basis for $RowA$.
- rank A = dimension of $ColA$ = the number of pivot columns = the number of pivot rows = dimension of $RowA$.
- nullity A = dimension of $NulA$.
- Suppose that A is of size $m \times n$, then
  - ∗ rank A + nullity A = n,
  - ∗ rank A + nullity $A^\top$ = m.

Exercise: Given a matrix $A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ -2 & 5 & -4 & 7 \\ 4 & 5 & -3 & 4 \end{bmatrix}$.

(1) Find a basis for $ColA$. $\left\{ \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix} \right\}$
(2) Does the columns of A span $\mathbb{R}^4$? No. Does the columns of A span $\mathbb{R}^3$? Yes.
(3) Find a basis for $RowA$. $\{[1 \quad 2 \quad 2 \quad 1], [-2 \quad 5 \quad -4 \quad 7], [4 \quad 5 \quad -3 \quad 4]\}$
(4) Does the rows of A span $\mathbb{R}^4$? No. Does the rows of A span $\mathbb{R}^3$? No.
(5) What is rank A? 3
(6) Find a basis for $NulA$. What is nullity A? 1
(7) Find a basis for $NulA^\top$. What is nullity $A^\top$? 0

3. **Coordinate system**:

Suppose that vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ and $\mathbf{x} \in V$. The coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$ (the $\mathcal{B}$-coordinate of $\mathbf{x}$) are the weights $c_1, c_2, \cdots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n,$$

and

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Exercise: Vector space $V = \mathbb{P}_2$ has a basis $\mathcal{B} = \{1, x, x^2\}$.

(1) What is the dimension of V? dim V=3

(2) Find the $\mathcal{B}$-coordinate of $p_1 = x^2 + x + 1$, $p_2 = x^2 - x + 1$, $p_3 = x^2$ that is, $[p_1]_{\mathcal{B}}$, $[p_2]_{\mathcal{B}}$, $[p_3]_{\mathcal{B}}$, respectively.

$$[p_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, [p_2]_{\mathcal{B}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} =, [p_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(3) Show that $\mathcal{C} = \{p_1, p_2, p_3\}$ is a basis for V.

(4) Find $p \in V$ such that $[p]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

$P = 3p_1 + 2p_2 - p_3 = 4x^2 + x + 5.$

4. **Change of basis:**

Suppose $\mathbb{R}$ has a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ and $\mathbf{x} \in \mathbb{R}^n$. Then

$$\mathbf{x} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n][\mathbf{x}]_{\mathcal{B}} = B[\mathbf{x}]_{\mathcal{B}},$$

where $B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n]$.

Suppose $\mathbb{R}$ has an another basis $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}$ and $\mathbf{x} \in \mathbb{R}^n$. Then

$$\mathbf{x} = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \cdots \quad \mathbf{c}_n][\mathbf{x}]_{\mathcal{C}}$$

In particular,

$$\mathbf{c}_1 = B[\mathbf{c}_1]_{\mathcal{B}}, \quad \mathbf{c}_2 = B[\mathbf{c}_2]_{\mathcal{B}}, \quad \cdots, \quad \mathbf{c}_n = B[\mathbf{c}_n]_{\mathcal{B}},$$

Hence

$$\mathbf{x} = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \cdots \quad \mathbf{c}_n][\mathbf{x}]_{\mathcal{C}} = B\Big[[\mathbf{c}_1]_{\mathcal{B}} \quad [\mathbf{c}_2]_{\mathcal{B}} \quad \cdots \quad [\mathbf{c}_n]_{\mathcal{B}}\Big][\mathbf{x}]_{\mathcal{C}}$$

Therefore,

$$[\mathbf{x}]_{\mathcal{B}} = \Big[[\mathbf{c}_1]_{\mathcal{B}} \quad [\mathbf{c}_2]_{\mathcal{B}} \quad \cdots \quad [\mathbf{c}_n]_{\mathcal{B}}\Big][\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}[\mathbf{x}]_{\mathcal{C}}$$

Exercise 4.1: Find the change of basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{C} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} \right\},$$

Sol: Do row reduction to

$$\begin{bmatrix} 0 & 3 & 7 & 2 & -2 & 5 \\ -2 & 5 & 1 & -3 & 1 & 4 \\ 4 & 3 & 1 & 1 & 3 & 0 \end{bmatrix}.$$

Exercise 4.2: Let $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ be basis of $\mathbb{R}^2$ such that $b_1 = 2c_1 + c_2, b_2 = c_1 + 2c_2$. Determine the $\mathcal{C}$-coordinate of $v = 2b_1 + 3b_2$.

Sol: $[v]_{\mathcal{C}} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$.