On the energy landscape of symmetric quantum signal processing

Jiasu Wang [1] Yulong Dong [1] Lin Lin [1,2,3]

[1]Department of Mathematics, University of California, Berkeley, CA 94720, USA
[2]Challenge Institute for Quantum Computation, University of California, Berkeley, CA 94720, USA
[3]Computational Research Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA

Symmetric quantum signal processing

Quantum signal processing (QSP) [1] finds a set of parameters (called phase factors) $\Phi := (\phi_0, \cdots, \phi_d) \in [-\pi, \pi)^{d+1}$, so that the real part of upper-left entry of the matrix $U(x, \Phi) \in \mathrm{SU}(2)$ gives a desired polynomial f:

$$f(x) = g(x, \Phi) := \mathrm{Re}[\langle 0|U(x, \Phi)|0\rangle], \quad x \in [-1, 1], \tag{1}$$

with

$$U(x, \Phi) := e^{\mathrm{i}\phi_0 Z} e^{\mathrm{i}\arccos(x) X} e^{\mathrm{i}\phi_1 Z} e^{\mathrm{i}\arccos(x) X} \cdots e^{\mathrm{i}\phi_{d-1} Z} e^{\mathrm{i}\arccos(x) X} e^{\mathrm{i}\phi_d Z}. \tag{2}$$

The SU(2) representation in Eqs. (1) and (2) can be directly translated into a quantum circuit for performing the following tasks:

- eigenvalue transformation $f(A)$ (when A is an Hermitian matrix) [1],
- singular value transformations $f^{\mathrm{SV}}(A)$ (when A is a general matrix) [2].

For any target function f satisfying (1) $\deg(f) = d$, (2) the parity of f is (d mod 2), (3) $\|f\|_\infty := \max_{x\in[-1,1]} |f(x)| < 1$, the solution to Eq. (1) exists [1, 2].

The number of degrees of freedom in f is only $\tilde{d} := \lceil \frac{d+1}{2} \rceil$. Ref. [3] suggests to be **symmetric**, i.e.

$$\Phi = (\underbrace{\phi_0, \phi_1, \phi_2, \ldots, \phi_{\tilde{d}-1}}_{\text{reduced phase factors } \tilde{\Phi}}, \ldots, \phi_2, \phi_1, \phi_0).$$

Optimization based method

The problem of finding phase factors can be reformulated as

$$\Phi^* = \underset{\substack{\Phi\in[-\pi,\pi)^{d+1},\\ \text{symmetric.}}}{\mathrm{argmin}}\ F(\Phi),\ F(\Phi) := \frac{1}{\tilde{d}} \sum_{k=1}^{\tilde{d}} |g(x_k, \Phi) - f(x_k)|^2, \tag{3}$$

where x_k are the positive nodes of the Chebyshev polynomial $T_{2\tilde{d}}(x)$.

Our works lie in:

- Characterize all the global minima of the cost function.
- Explain this phenomenon: starting from a fixed initial guess, the solution can always be robustly obtained in practice (even for d as large as 10^4) [3].

Existence and uniqueness of symmetric phase factors

Theorem 1 Consider any $P \in \mathbb{C}[x]$ and $Q \in \mathbb{R}[x]$ satisfying the following conditions,

1. $\deg(P) = d$ and $\deg(Q) = d - 1$.
2. P has parity (d mod 2) and Q has parity ($d - 1$ mod 2).
3. (Normalization condition) $\forall x \in [-1, 1] : |P(x)|^2 + (1 - x^2)|Q(x)|^2 = 1$.
4. If d is odd, then the leading coefficient of Q is positive.

There exists a unique set of symmetric phase factors $\Phi \in D_d$ such that

$$U(x, \Phi) = \begin{pmatrix} P(x) & \mathrm{i}Q(x)\sqrt{1-x^2} \\ \mathrm{i}Q(x)\sqrt{1-x^2} & P^*(x) \end{pmatrix}, \tag{4}$$

where

$$D_d = \begin{cases} [-\frac{\pi}{2}, \frac{\pi}{2})^{\frac{d}{2}} \times [-\pi, \pi) \times [-\frac{\pi}{2}, \frac{\pi}{2})^{\frac{d}{2}}, & d \text{ is even}, \\ [-\frac{\pi}{2}, \frac{\pi}{2})^{d+1}, & d \text{ is odd}. \end{cases} \tag{5}$$

Global minima

(P, Q) is said to be an **admissible pair of polynomials** associated with f if the conditions in Theorem 1 are satisfied and the leading coefficient of Q is **positive**.

- If d is odd, there is a bijection between the global minima of Eq. (3) and all admissible pairs (P, Q).
- If d is even, there is a bijection between the global minima of Eq. (3) and all pairs of polynomials $(P, \pm Q)$, where (P, Q) is an admissible pair.

Construction of admissible pairs

Theorem 2 Given a target polynomial $f(x)$, all admissible pairs (P, Q) must take the following form,

$$\begin{aligned} \mathrm{Im}[P](x) &= \sqrt{\alpha} \frac{e\left(x + \mathrm{i}\sqrt{1-x^2}\right) + e\left(x - \mathrm{i}\sqrt{1-x^2}\right)}{2}, \\ Q(x) &= \sqrt{\alpha} \frac{e\left(x + \mathrm{i}\sqrt{1-x^2}\right) - e\left(x - \mathrm{i}\sqrt{1-x^2}\right)}{2\mathrm{i}\sqrt{1-x^2}}, \end{aligned} \tag{6}$$

where

$$\mathfrak{F}(z) := 1 - \left[f\left(\frac{z + z^{-1}}{2}\right)\right]^2 = \alpha e(z) e(z^{-1}), \quad e(z) := z^{-d} \prod_{i=1}^{2d} (z - r_i). \tag{7}$$

The value of $\alpha \in \mathbb{C}$ depends on $\{r_i\}_{i=1}^{2d}$, which are roots of the *Laurent polynomial* $\mathfrak{F}(z)$,

All the admissible pairs can be constructed by properly choosing $\{r_i\}_{i=1}^{2d}$ such that it is closed under additive inverse and complex conjugate.

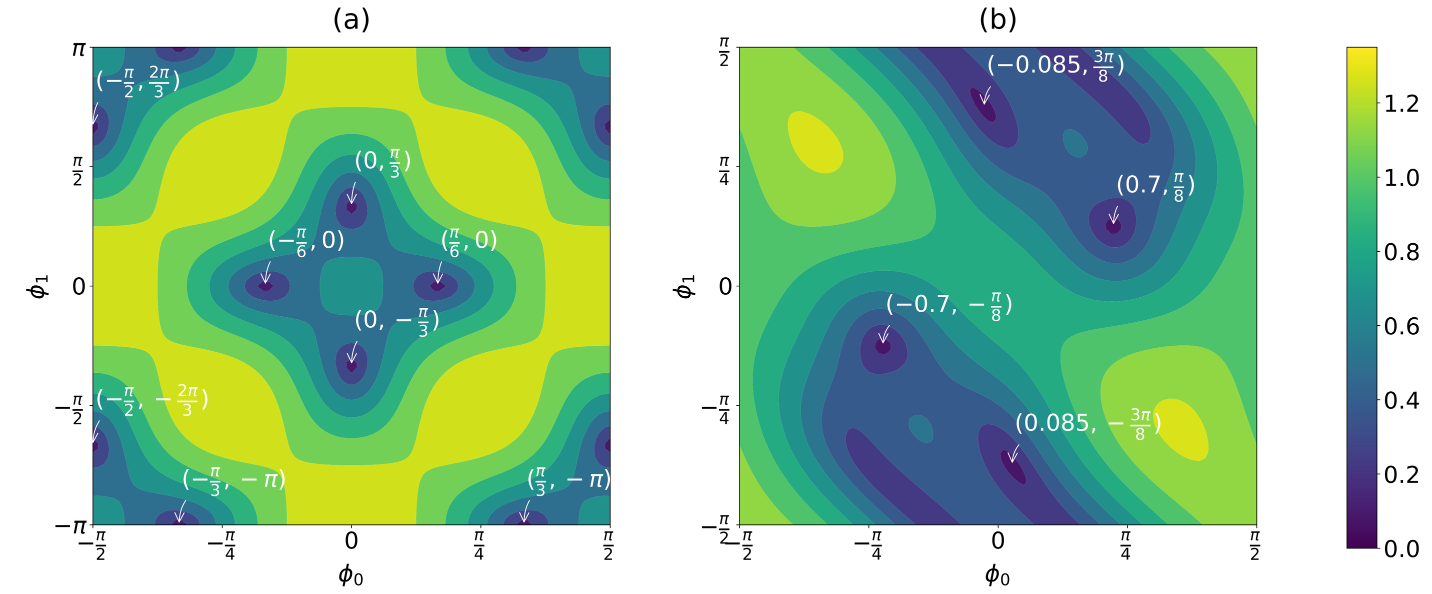

Figure 1. The optimization landscape of the modified cost function $F(\tilde{\Phi})^{1/3}$ over D_d. (a) $f(x) = x^2 - \frac{1}{2}$. (b) $f(x) = \frac{1}{\sqrt{3}}x^3 - \frac{2}{\sqrt{3}}x$.

Existence of local minima can be easily observed when $\deg(f) \geq 4$.

Maximal solution

Choose $\{r_i\}_{i=1}^{2d}$ to be the roots of $\mathfrak{F}(z)$ within the unit disc. The unique symmetric phase factor associated with this admissible pair is called the **maximal solution**.

For any target polynomial f with $\|f\|_\infty \leq \frac{1}{2}$, the maximal solution lies in the neighborhood of $\Phi^0 := (\pi/4, 0, 0, \ldots, 0, 0, \pi/4)$. Φ^0 is the initial guess used for the optimization method [3].

Distance between the maximal solution and Φ^0

Theorem 3 Let Φ^* be the maximal solution for the target function f. Denote $\tilde{\Phi}^*$ and $\tilde{\Phi}^0$ as the corresponding reduced phase factors of Φ^* and Φ^0 respectively. If $\|f\|_\infty \leq \frac{1}{2}$, then

$$\left\|\tilde{\Phi}^* - \tilde{\Phi}^0\right\|_2 \leq \frac{\pi}{\sqrt{3}} \|f(x)\|_\infty. \tag{8}$$

Local strong convexity

Theorem 4 If the target polynomial satisfies $\|f\|_\infty \leq \frac{\sqrt{3}}{20\pi d}$, for any symmetric phase factors Φ of length $d+1$ satisfying $\left\|\tilde{\Phi} - \tilde{\Phi}^0\right\|_2 \leq \frac{1}{20d}$, the following estimate holds:

$$\frac{1}{4} \leq \lambda_{\min}\left(\mathrm{Hess}(\tilde{\Phi})\right) \leq \lambda_{\max}\left(\mathrm{Hess}(\tilde{\Phi})\right) \leq \frac{25}{4}. \tag{9}$$

When $\|f\|_\infty$ is small enough, projected gradient method can converge exponentially in a neighborhood of Φ^0.

- $\mathcal{O}(\log(1/\epsilon))$ iterations, independent of any further details of f.
- Only use standard double precision arithmetic operations in practice.

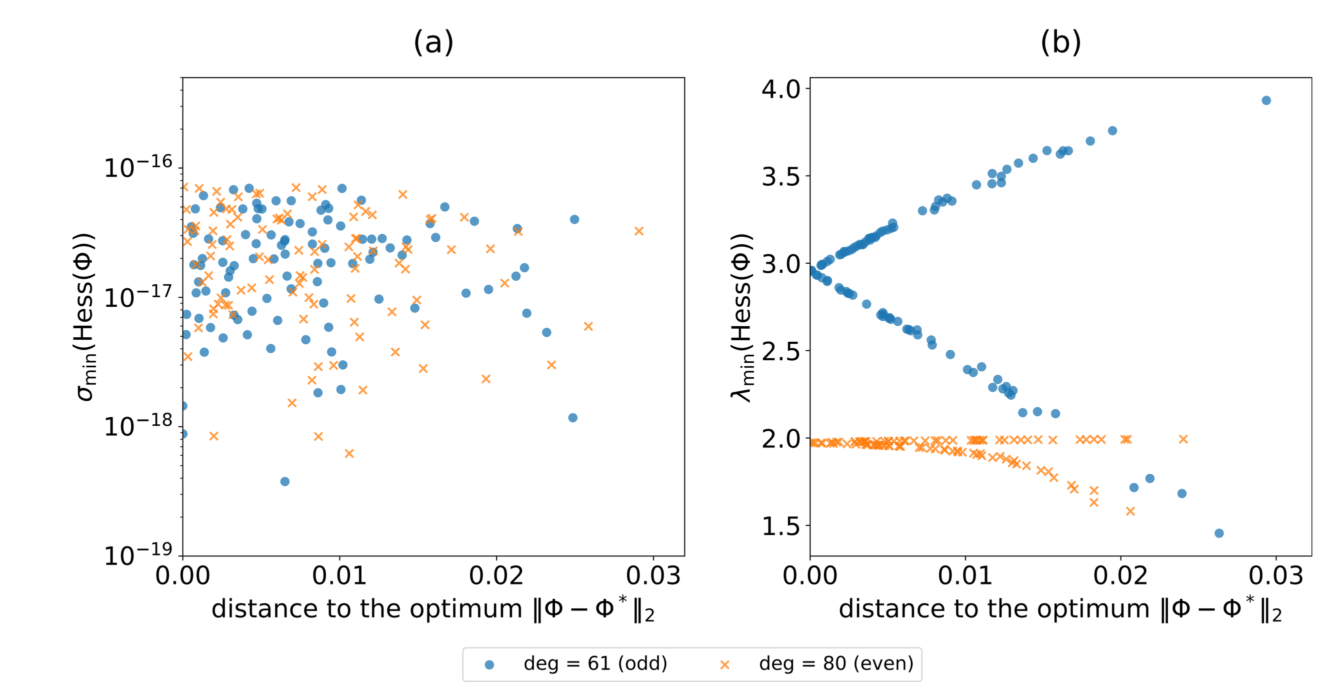

Figure 2. The smallest singular value ($\sigma_{\min}$) or eigenvalue ($\lambda_{\min}$) of the Hessian matrix evaluated at 100 randomly sampled Φ near the optimizer Φ^*. (a) Without symmetry constraint. (b) With symmetry constraint.

References and acknowledgements

[1] Guang Hao Low and Isaac L Chuang. Optimal hamiltonian simulation by quantum signal processing. *Physical review letters*, 118(1):010501, 2017.

[2] András Gilyén, Yuan Su, Guang Hao Low, and Nathan Wiebe. Quantum singular value transformation and beyond: exponential improvements for quantum matrix arithmetics. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, pages 193–204. ACM, 2019.

[3] Yulong Dong, Xiang Meng, K. Birgitta Whaley, and Lin Lin. Efficient phase-factor evaluation in quantum signal processing. *Physical Review A*, 103(4), Apr 2021.

[4] Jiasu Wang, Yulong Dong, and Lin Lin. On the energy landscape of symmetric quantum signal processing. *arXiv:2110.04993*, 2021.

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