Quantum-accelerated multilevel Monte Carlo methods

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Problem setting

Given a stochastic differential equation (SDE)

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sigma(X_t, t)\mathrm{d}W_t, \tag{1}$$

we aim to compute

$$\mathbb{E}[\mathcal{P}(X_T)|X_0 \in \pi_0]. \tag{2}$$

- X_0 follows a initial distribution π_0,
- T is evolution time,
- $\mathcal{P}(X)$ is the payoff function.

Some assumptions:

- μ and σ are globally Lipschitz continuous,
- $\mathcal{P}$ is piecewise Lipschitz continuous.

Classical Monte Carlo method

The Milstein discretization of strong order 1

$$\begin{aligned}\widehat{X}_{k+1} =& \widehat{X}_k + \mu(\widehat{X}_k, t_k)h + \sigma(\widehat{X}_k, t_k)\Delta W_k \\ &+ \frac{1}{2}\sigma(\widehat{X}_k, t_k)\partial_X \sigma(\widehat{X}_k, t_k)((\Delta W_k)^2 - h).\end{aligned}$$

To achieve a mean-squared error with bound $\mathbb{E}(Y - \mathbb{E}[P])^2 \leq \epsilon^2$,

- $\widehat{X}_n$ approximates X_T with $\mathbb{E}|\widehat{X}_n - X_T| = O(h)$,
- the number of iterations is $n = T/h = \Omega(1/\epsilon)$,
- the number of samples to estimate $\mathbb{E}[\mathcal{P}(X_T)]$ is $N = O(1/\epsilon^2)$,

the computational cost is $O(N/h) = O(1/\epsilon^3)$.

Solve SDE

Apply a numerical scheme of strong order r $\left(\mathbb{E}|\widehat{X}_n - X_T| = O(h^r)\right)$.

Theorem 1

- *Classical Monte Carlo method estimates* $\mathbb{E}[\mathcal{P}(X_T)]$ *up to mean-squared error* ϵ^2 *in cost* $O(\epsilon^{-2-1/r})$.

- *Quantum-accelerated Monte Carlo method (QA-MC) estimates* $\mathbb{E}[\mathcal{P}(X_T)]$ *up to additive error* ϵ *with probability at least 0.99 in cost* $O(\epsilon^{-1-1/r})$.

Multilevel Monte Carlo method

Given a sequence $P_0, P_1, \ldots, P_L$ that approximates P

- V_l: the variance of one sample of $P_l - P_{l-1}$ decreases,
- C_l: the cost of one sample of $P_l - P_{l-1}$ increases,
- N_l: the number of samples of $P_l - P_{l-1}$.

We use this estimator to to approximate $\mathbb{E}[P]$,

$$Y = \sum_{l=0}^{L} Y_l, \quad \text{where } Y_l := \frac{1}{N_l} \sum_{i=0}^{N_l} \left(P_l^{(l,i)} - P_{l-1}^{(l,i)} \right). \tag{3}$$

For SDE, $P = \mathcal{P}(X_T)$ and $P_l - P_{l-1}$ comes from two discrete approximations with different timesteps but the same Brownian path.

Multilevel Monte Carlo method

The overall cost and variance of Y is $\sum_{l=0}^{L} N_l C_l$ and $\sum_{l=0}^{L} N_l^{-1} V_l$. To trade off between cost and variance, we consider

$$\min_{N_l, l=0,\cdots,L} \sum_{l=0}^{L} N_l C_l + \lambda^2 \left(\frac{\epsilon^2}{2} - \sum_{l=0}^{L} N_l^{-1} V_l \right). \tag{4}$$

- $N_l = \lambda \sqrt{V_l / C_l}$ with $\lambda = 2\epsilon^{-2} \sum_{l=0}^{L} \sqrt{V_l C_l}$,
- The total computational cost is

$$2\epsilon^{-2} \left(\sum_{l=0}^{L} \sqrt{V_l C_l} \right)^2, \tag{5}$$

to achieve $\mathbb{E}(Y - \mathbb{E}[P])^2 \leq \epsilon^2$.

Classical multilevel Monte Carlo method

If there exist positive constants α, β, γ such that $\alpha \geq \frac{1}{2}\min(\beta, \gamma)$ and

- $|\mathbb{E}[P_l - P]| = O(2^{-\alpha l})$,
- $V_l = O(2^{-\beta l})$,
- $C_l = O(2^{\gamma l})$,

then for any $\epsilon < 1/e$ there exists an L such that $Y = \sum_{l=0}^{L} Y_l$ has a mean-squared error with bound $\mathbb{E}(Y - \mathbb{E}[P])^2 \leq \epsilon^2$.
Moreover, the total computational cost is

$$\begin{cases} O(\epsilon^{-2}), & \beta \geq \gamma, \\ O(\epsilon^{-2-(\gamma-\beta)/\alpha}), & \beta < \gamma. \end{cases} \tag{6}$$

Quantum-accelerated multilevel Monte Carlo method

Theorem 2 (QA-MLMC)

If there exist positive constants $\alpha, \beta = 2\hat{\beta}, \gamma$ *such that* $\alpha \geq \min(\hat{\beta}, \gamma)$ *and*

- $|\mathbb{E}[P_l - P]| = O(2^{-\alpha l})$,
- $V_l = O(2^{-\beta l}) = O(2^{-2\hat{\beta} l})$,
- $C_l = O(2^{\gamma l})$,

then for any $\epsilon < 1/e$ *there is a quantum algorithm that estimates* $\mathbb{E}[P]$ *up to additive error* ϵ *with probability at least 0.99, and with cost*

$$\begin{cases} O\left(\epsilon^{-1}\right), & \hat{\beta} \geq \gamma, \\ O\left(\epsilon^{-1-(\gamma-\hat{\beta})/\alpha}\right), & \hat{\beta} < \gamma. \end{cases} \tag{7}$$

Solve SDE

Using a numerical scheme of strong order r,

$$\alpha = r - o(1), \quad \beta = r - o(1), \text{ and } \gamma = 1. \tag{8}$$

Theorem 3

- *MLMC estimates* $\mathbb{E}[\mathcal{P}(X_T)]$ *up to mean-squared error* ϵ^2 *in cost*

$$\begin{cases} O(\epsilon^{-2}), & r > 1, \\ O(\epsilon^{-1-1/r-o(1)}), & r \leq 1. \end{cases} \tag{9}$$

- *QA-MLMC estimates* $\mathbb{E}[\mathcal{P}(X_T)]$ *up to additive error* ϵ *with probability at least 0.99 in cost*

$$\begin{cases} O\left(\epsilon^{-1}\right), & r > 2, \\ O\left(\epsilon^{-1/2-1/r-o(1)}\right), & r \leq 2. \end{cases} \tag{10}$$

Applications

- Black-Scholes model:
  The asset price is modelled by Geometric Brownian Motion (GBM)

  $$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t,$$

  where σ measures the volatility of the asset and μ is the drift rate.
- Local Volatility model:
  It generalizes GBM as

  $$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma(X_t, t) X_t \mathrm{d}W_t,$$

  by treating volatility σ as a function of the asset X_t and the time t.

Applications

Option pricing

- Lipschitz continuous options
  - European option: $\mathcal{P}(X_T) = (X_T - K)^+ := \max\{X_T - K, 0\}$,
  - Asian option: $\mathcal{P}(X_T) = \left(\frac{1}{T}\int_0^T X_t \mathrm{d}t - K\right)^+$.
- Piecewise Lipschitz continuous options
  - Digital option

$$\mathcal{P}(X_T) = \mathcal{H}(X_T - K),$$

with the strike $K > 0$, where $\mathcal{H}$ is the Heaviside function.

Other applications:

Greeks (sensitivity of price), Binomial option pricing model, ...

Summary

	Algorithm	Model	Result
Classical	MC with direct sampling	Black-Scholes model	ϵ^{-2}
	MC with scheme of strong order r	payoff models of general SDEs	$\epsilon^{-2-1/r}$
	MLMC with scheme of strong order $r > 1$	payoff models of general SDEs	ϵ^{-2}
	MC with binomial sampling	binomial option pricing model	ϵ^{-2}
Quantum	QA-MC with direct sampling	Black-Scholes model	ϵ^{-1}
	QA-MC with scheme of strong order r	payoff models of general SDEs	$\epsilon^{-1-1/r}$
	QA-MLMC with scheme of strong order $r > 2$	payoff models of general SDEs	ϵ^{-1}
	QA-MC with binomial sampling	binomial option pricing model	ϵ^{-1}

Table: Summary of the time complexities of classical and quantum algorithms for financial models with the additive error ϵ, in which logarithmic factors are omitted.